Department of Mathematics
University of Louisiana at Lafayette

Numerical Analysis Comprehensive Examination

Friday, August 15, 2014, 9:00AM to 1:00PM, MDD 212

This exam is closed book. You may use Matlab, but you may not consult the internet for material or answers. Also, this should be your own work: you may not consult with others taking the exam. These rules will be enforced. Please use your own paper, and put you name on each sheet.

1. Consider evaluating
$$f(x) = \sinh(x) = (e^x - e^{-x})/2$$
for $x \in [-10^{-16}, 10^{-16}]$.
   (a) Evaluate $f(10^{-16})$ and store the result using Matlab's `sinh` function.
   (b) Evaluate $f(10^{-16})$ using the expression $f(x) = (e^x - e^{-x})/2$ and store the result.
   (c) Evaluate $f(10^{-16})$ using the degree 3 Taylor polynomial for sinh centered at $x = 0$.
   (d) Assuming you know the decimal representation of the closest floating point number to $\sinh(10^{-16})$ is 1×10^{-16}, use this to compute an approximation to the relative error in the computations in parts 1a, 1b, and 1c.
   (e) State your conclusions from part 1d, and explain why you observe the relative error you do.
   (f) Print and hand in your Matlab computations along with your explanation. (Put your name inside any files you print.)

2. Consider the function $g(x) = e^{-x}$.
   (a) Show that $g(x)$ has a unique fixed point $z \in (-\infty, \infty)$.
   (b) Prove that g is a contraction on $[\ln 1.1, \ln 3]$.
   (c) Prove that $g : [\ln 1.1, \ln 3] \to [\ln 1.1, \ln 3]$.
   (d) Prove that $x_{k+1} = g(x_k)$ converges to the unique fixed point $z \in (-\infty, \infty)$ for *any* $x_0 \in (-\infty, \infty)$.

3. Define an $n \times n$ real matrix A by $a_{ij} = w_j^T w_i$ for n real vectors $w_1, w_2, \ldots, w_n$.
   (a) Prove that A is real, symmetric, and positive semi-definite.
   (b) Under what condition is A positive definite?
   (c) Show that there exists a spectral decomposition ($A = VDV^T$) and a singular value decomposition that are identical.

4. Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 1.5 \\ 1 & 2.1 \\ 1 & 3 \end{pmatrix}$ and $b = \begin{pmatrix} 2.11 \\ 6.43 \\ 8.1 \\ 11.14 \end{pmatrix}$.
   (a) Compute a singular value decomposition of A. (You may use a tool such as Matlab, but show each step.)
   (b) Use the singular value decomposition to compute the least squares solution to $Ax = b$.

Your computations do not need to be mathematically rigorous, that is, they do not need to take account of small roundoff errors, but you should hand in all steps (such as a copy of a Matlab dialog, plus a thorough explanation).

5. Find the least squares approximation $p_1(x) = a_0 + a_1 x$ of the function

$$f(x) = \frac{1}{1+x^2}$$

with respect to the norm

$$\|f - p\| = \sqrt{\int_{-1}^{1} (f(x) - p(x))^2 dx.}$$

State both the coefficients a_0 and a_1 and the minimum residual $\|f - p_1\|$.

6. Consider the power method for computing eigenvalues and eigenvectors of a matrix A.
   (a) Explain fully the principle behind the power method. Use formulas in your explanation, as appropriate.
   (b) Suppose the matrix A has two eigenvalues, equal to $\lambda_1 \approx 1$ and $\lambda_2 \approx 0.1$. The convergence rate of the power method to the eigenvector v_1 corresponding to λ_1 is linear; what is the approximate convergence rate?
   (c) How would you modify the matrix A so the power method will converge in a way that you could recover the eigenvalue λ_2 and corresponding eigenvector v_2? Explain fully.
   (d) Explain how to get the complete set of eigenvalues of A, including the main steps of the algorithm and the computational complexity of each step.

7. Recall that Simpson's rule is given by

$$\int_{-1}^{1} f(x)dx = \frac{1}{3}f(-1) + \frac{4}{3}f(0) + \frac{1}{3}f(1) - \frac{1}{90}f^{(iv)}(c)$$

for some $c \in [-1, 1]$. Use this to compute both the Simpson's rule approximation to

$$\int_{0}^{0.1} x \sin(x) dx$$

and a bound on the error in the approximation. Give the approximation and the bound on the error to 6 significant digits.
Hint: The fourth derivative of $x \sin(x)$ is $-4\cos(x) + x\sin(x)$.

8. Find the region of absolute stability in the complex plane for the implicit trapezoidal rule

$$y_{k+1} = y_k + h\frac{f(t_k, y_k) + f(t_{k+1}, y_{k+1})}{2}.$$

Draw a diagram of the region of stability and describe it in words.

9. Rigorously prove there are no solutions to $F(x) = (x_1 - \sin(x_2), x_2^2 - x_1^2)^T$ within the box $\boldsymbol{x} = ([1, 2], [0, 1.5])$. *(You may use the computer and interval arithmetic, but you are not required to do so. In any case, provide a full and detailed explanation.)*