Department of Mathematics
University of Louisiana at Lafayette

Numerical Analysis Comprehensive Examination

Tuesday, August 11, 2009, 9:00AM to 1:00PM, MDD 208

This exam is closed book, but calculators are permitted. Please use your own paper, and put you name on each sheet.

1. Write down a polynomial $p(x)$ such that $|\mathrm{erf}(x) - p(x)| \le 0.01$ for $-0.1 \le x \le 0.1$, where
$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$
is the *error function* (well-known in statistics, etc.). Prove that your polynomial p satisfies the condition $|\mathrm{erf}(x) - p(x)| \le 0.01$ for $x \in [-0.1, 0.1]$.
(*At your discretion, you may represent the polynomial's coefficients in symbolic form, as approximations to four correct significant decimal digits, or as rigorous interval enclosures. To bound the error, you may, but don't necessarily need, to use facts about alternating series.*)

2. Assume that the equation $x^2 + bx + c = 0$ has two real roots α and β with $|\alpha| < |\beta|$. Show that the iteration method
$$x_{k+1} = -\frac{c}{x_k + b}$$
is convergent to α if x_0 is sufficiently close to α.

3. Consider Euler's method for approximating the IVP $y'(x) = f(x, y), 0 < x < a, y(0) = \alpha$. Let $y_h(x_{i+1}) = y_h(x_i) + hf(x_i, y_h(x_i))$ for $i = 0, 1, \ldots, N$ where $y_h(0) = \alpha$. It is known that $y_h(x_i) - y(x_i) = c_1 h + c_2 h^2 + c_3 h^3 + \ldots$ where $c_m, m = 1, 2, 3, \ldots$ depend on x_i but not on h. Suppose that $y_h(a), y_{\frac{h}{2}}(a), y_{\frac{h}{3}}(a)$ have been calculated using interval width: $h, \frac{h}{2}, \frac{h}{3}$, respectively. Find an approximation $\hat{y}(a)$ to $y(a)$ that is accurate to order h^3.

4. Derive the normal equations, used in theoretical analysis of least squares approximation, from the least squares condition
$$\frac{1}{2}\|Ax^* - b\|_2^2 = \min_{x \in \mathbb{R}^n} \varphi(x), \quad \text{where} \;\; \varphi(x) = \frac{1}{2}\|Ax - b\|_2^2.$$

5. Suppose $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 3 \end{pmatrix}$, and $A = U\Sigma V^T$, where U and V are orthogonal,
$$U \approx \begin{pmatrix} -0.5475 & -0.1832 & -0.8165 \\ -0.3231 & -0.8538 & 0.4082 \\ -0.7719 & 0.4873 & 0.4082 \end{pmatrix},$$
$$\Sigma \approx \begin{pmatrix} 4.0791 & 0 \\ 0 & 0.6005 \\ 0 & 0 \end{pmatrix}, \text{ and}$$
$$V \approx \begin{pmatrix} -0.4027 & -0.9153 \\ -0.9153 & 0.4027 \end{pmatrix}.$$
Compute an approximation to the least squares solution to $Ax = b$, where $b = (1, -1)^T$.

6. Find the least squares approximation for the function $f(x) = x^3$ by a polynomial $p_1(x) = b_0 + b_1 x$ on $[-1, 1]$ using the norm

$$||f||^2 = \int_{-1}^{1} (f(x))^2 dx.$$

7. Given that

$$\int_{-1}^{1} f(x)dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) + \frac{1}{135} f^{(4)}(\theta),$$

for some $\theta \in [-1, 1]$, find an interval enclosure to $\log(1.1)$. Use the above integration formula, rather than other knowledge.

Hint: You can make a change of variables from $[-1, 1]$ *to* $[1, 1.1]$.

8. Find the region of absolute stability for the trapezoidal method:

$$y_{j+1} = y_j + \frac{h}{2}\left(f(t_j, y_j) + f(t_{j+1}, y_{j+1})\right), \quad j = 0, 1, \cdots, N-1.$$

9. Consider $x^{(k+1)} = \begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \end{bmatrix} = \begin{bmatrix} 0.5 & -0.25 \\ -0.25 & 0.5 \end{bmatrix} \begin{bmatrix} \cos(x_1^{(k)}) \\ \sin(x_2^{(k)}) \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, with $x^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Prove that $\{x^{(k)}\}_{k=0}^{\infty}$ converges to a unique $x^* \in \mathbb{R}^2$.

10. Prove that $F(x) = 0$ has a unique solution in $([2.4, 2.6], [1.5, 1.6])^T$, where

$$F(x) = \begin{pmatrix} \sin(x_1) - x_2 + 1 \\ x_1 - x_2^2 \end{pmatrix}.$$