Numerical Analysis Comprehensive Examination

Thursday, August 10, 2006, 9:00AM–1:00PM, MDD 208

Instructions: Present all work carefully and completely. This exam is to be done without the aid of the text and notes, although calculators are permissible.

1. Suppose that f has continuous derivatives.

   (a) Show that

   $$\frac{d^4 f}{dx^4}(x) = \frac{1}{h^4}\Big\{f(x+2h) - 4f(x+h) + 6f(x) - 4f(x-h) + f(x-2h)\Big\} + \mathcal{O}(h^2).$$

   (b) How many continuous derivatives are necessary for the above approximation to hold?

2. Use interval arithmetic with either exact endpoints or with three-significant-digit outward rounding to prove that $f(x) = x^3 - x + 1 = 0$ has no solutions for $x \in [1/2, 1]$.

3. Treating Newton's method as a fixed-point iteration, show that Newton's method starting with any $x_0 \in [1.4, 1.6]$, applied to $f(x) = x^2 - 2$, will converge.

4. Let A be nine-diagonal, that is, let A have non-zeros only on the diagonal, the four entries above the diagonal, and the four entries below the diagonal. (See figure 1.) Assuming that

   $$A = \begin{pmatrix} * & * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & * & 0 \\ * & * & * & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * & * \end{pmatrix}.$$

   Figure 1: A nine-diagonal matrix

   Gaussian elimination without pivoting is numerically stable for the nine-diagonal linear systems we wish to solve, write down an expression for the number of operations required to complete the factorization (i.e. the forward) phase of Gaussian elimination on the matrix A, as a function of the order n of A.

5. Suppose

   $$A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -1.0065 \\ 0.9917 \\ 2.9410 \end{pmatrix},$$

   and $A = QR$, where

   $$Q \approx \begin{pmatrix} -0.5774 & 0.7071 & 0.4082 \\ -0.5774 & 0.0000 & -0.8165 \\ -0.5774 & -0.7071 & 0.4082 \end{pmatrix}, \quad \text{and} \quad R \approx \begin{pmatrix} -1.7321 & 0 \\ 0 & -1.4142 \\ 0 & 0 \end{pmatrix},$$

where you may assume that Q is orthogonal. Using this information, compute an approximate least squares solution to $Ax = b$. Show all steps.

6. Compute a bound on the error in interpolating $f(x) = \sin(x)$ for $x \in [-0.1, 0.1]$ by a polynomial that interpolates at the points -0.1, 0, and 0.1. Make the error bound as small as you can, subject to it being an actual bound. Show all of your work.

7. Consider
$$A = \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}.$$
   (a) Compute the eigenvalues and eigenvectors of A directly from the characteristic equation.
   (b) Apply several iterations of the power method, starting with initial guess vector $(1, 1)^T$. Explain your results in terms of the exact eigenvalues and eigenvectors you computed in the previous step.

8. Show that the single step method
$$y_{j+1} = y_j + f\left(t_j + \frac{h}{2}, y_j + \frac{h}{2} f(t_j, y_j)\right)$$
for solving the initial value problem
$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0$$
has order of accuracy $p = 2$.

9. Do several iterations of the multivariate Newton method applied to find a solution of $F(x) = 0$, where
$$F(x) = (x_1^2 + x_2^2 - 1, x_1 + x_2)^T, \quad \text{with} \quad x^{(0)} = (1, 0)^T.$$
Compare the result with the result you obtain by solving the system of equations exactly.