Math. 556-01
Spring, 1998
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**Final Examination**

*Take-home, due before Monday, May 11, 1998, 1:00 PM*

**Instructions:** This exam may be done outside of class at your convenience. You may consult any appropriate references. However, you may not collaborate in any way with others in the class on this particular work. The references have been placed on two hour reserve in Dupré Library.

1. Let
$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \tag{1}$$
and let $A = L + U + D$, where $L$, $U$, and $D$ are the lower triangular, upper triangular, and diagonal parts of $A$, as in the Gauss–Seidel method.
   (a) Write down explicit numerical approximations for the eigenvalues of the iteration matrix $G_{\text{J}} = D^{-1}(L+U)$ for the Jacobi method.
   (b) Write down explicit numerical approximations for the eigenvalues of the iteration matrix $G_{\text{GS}} = (D+L)^{-1}U$ for the Gauss–Seidel method. Compare.
   (c) Let $A$ be a general matrix, and let $G_{\text{J}}$ and $G_{\text{GS}}$ denote the corresponding iteration matrices for the Jacobi and Gauss–Seidel methods, respectively. Can you say anything about the relationship of the spectral radii $\rho(G_{\text{J}})$ and $\rho(G_{\text{GS}})$? (*Hint: You may look at the references.*) Are there additional conditions on $A$ (such as symmetry or positive-definiteness) that will allow you to say more?

2. Consider the ordinary differential equation
$$y' = f(x,y), \quad y(x_0) = y_0.$$
In class, we discussed Euler's method, and analyzed its stability for the model problem $y' = -ay$, $y(0) = 1$, $a > 0$ some constant. Consider now a second-order Runge–Kutta method of the form
$$v^{(i+1)} = v^{(i)} + hf\left(x_i + \frac{h}{2}, v^{(i)} + \frac{h}{2} f(x_i, v^{(i)})\right)$$

(known as the "midpoint method").

(a) Determine an interval of stability for the midpoint method, for the model problem, using the technique that was presented in class. Compare with the corresponding region of stability from Euler's method and the implicit Euler's method. Do order and stability necessarily have a relationship?

(b) Show that the local order of the midpoint method is 2, where the order of the method is defined in Definition 10.7, page 516 of the Neumaier manuscript. Is there necessarily a relationship between order and stability?

3. Consider integrals of the form

$$I(f) = \int_0^1 \sqrt{x} f(x) dx.$$

(a) Compute $w_1$, $w_2$, $x_1$, and $x_2$ so

$$I(f) \approx w_1 f(x_1) + w_2 f(x_2)$$

is exact when $f$ is a polynomial of degree three or less.

(b) Write down the error term $E(f)$ in terms of a higher-order derivative of $f$, where

$$I(f) = w_1 f(x_1) + w_2 f(x_2) + E(f).$$

Prove that this error term is valid. (*Hint: You may wish to consult the relevant sections of [3].*)

4. Consider the *nonlinear* boundary value problem

$$-u''(x) = f(u), \quad u(0) = u(1) = 0,$$

with $f(u) = u^3$. Consider a grid with $h = 1/4$ (three internal nodes).

(a) Discretize the problem using central differences.

(b) Discretize the problem with the finite element method using hat functions. Do you obtain the same nonlinear system of three equations in three unknowns?

(c) Find all solutions to the systems of three equations in three unknowns in parts 4a and 4b, within the initial region $-10 \leq u_i \leq 10$, $1 \leq i \leq 3$. (*Hint: You may use the program* `run_roots_delete`, *an improved version of that described in [5, Ch. 4]. This program will be described in class.*)

Note: This problem was considered as an example in [4].

## References


[1] G. H. Golub and C. F. Van Loan. *Matrix Computations (Second Edition).* Johns Hopkins University Press, Baltimore, MD, 1989.

[2] L. A. Hageman and Young D. M. *Applied Iterative Methods.* Academic Press, 1981.

[3] R. W. Hamming. *Numerical Methods for Scientists and Engineers, second edition.* McGraw-Hill, New York, etc., 1973.

[4] R. B. Kearfott. Some general bifurcation techniques. *SIAM J. Sci. Statist. Comput.*, 4(1):52–68, March 1983.

[5] R. B. Kearfott. *Rigorous Global Search: Continuous Problems.* Kluwer, Dordrecht, Netherlands, 1996.

[6] John R. Rice. *Numerical Methods, Software, and Analysis.* Mc-Graw-Hill, New York, 1983.

[7] David M. Young. *Iterative Solution of Large Linear Systems.* Academic Press, New York, 1971.