1. We obtain a basis by row reduction:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 2 & 3 & 1 \\ 2 & 5 & 4 & 4 \end{bmatrix} \underset{\substack{R_3 \leftarrow R_3 - R_1 \\ R_4 \leftarrow R_4 - 2R_1}}{\overset{R_2 \leftarrow R_2 + R_1}{\sim}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 3 & 2 & 2 \end{bmatrix}$$

$$\overset{R_2 \leftarrow R_2/2}{\sim} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 3 & 2 & 2 \end{bmatrix} \underset{R_4 \leftarrow R_4 - 3R_2}{\overset{R_3 \leftarrow R_3 - R_2}{\sim}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 2 & -1 \end{bmatrix}$$

$$\overset{R_4 \leftarrow R_4 - R_3}{\sim} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence, a basis for the row space of A is the set of vectors $\{(1,1,1,1), (0,1,0,1), (0,0,2,-1)\}$ since they are linearly independent and span the space.

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2. Since $A \in \mathbb{R}^{4\times 4}$ and the rank of A is 3, the null space is non-trivial. We can obtain the null space of A from the reduced row echelon form for A. We continue the computations from problem 1:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{R_3 \leftarrow R_3/2}{\sim} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{R_1 \leftarrow R_1 - R_3}{\sim} \begin{bmatrix} 1 & 1 & 0 & 3/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overset{R_1 \leftarrow R_1 - R_2}{\sim} \begin{bmatrix} 1 & 0 & 0 & +1/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, solving for $x_1, x_2,$ and x_3 in terms of x_4 gives:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -1/2 \\ -1 \\ 1/2 \\ 1 \end{bmatrix},$$

and a basis for the null space is

$$\left\{ \begin{bmatrix} -1/2 \\ -1 \\ 1/2 \\ 1 \end{bmatrix} \right\}.$$

3 (a) The rank of A is 3, since that is the dimension of the row space.

(b) The dimension of the null space is 1, since the null space is spanned by a single vector.

(c) The dimension of the column space is 3, since the dimension of the column space is the same as the dimension of the row space.

4 The vector lies in the row space if and only if it is a linear combination of basis vectors for the row space. Since the nonzero ~~vectors~~ rows in any echelon form for A form a basis for the row space, the vector is in the row space if and only if there is a solution to the following system:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \\ 1 \end{bmatrix}.$$

We check this with our usual row reduction procedure.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 2 & 5 \\ 1 & 1 & -1 & 1 \end{array}\right] \underset{\substack{R_3 \leftarrow R_3 - R_1 \\ R_4 \leftarrow R_4 - R_1}}{\overset{R_2 \leftarrow R_2 - R_1}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 2 \\ 0 & 1 & -1 & -2 \end{array}\right] \overset{R_4 \leftarrow R_4 - R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & -1 & -1 \end{array}\right]$$

$$\overset{R_3 \leftarrow R_3/2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{array}\right] \overset{R_4 \leftarrow R_4 + R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, the system is consistent and $\begin{bmatrix} 3 \\ 2 \\ 5 \\ 1 \end{bmatrix}^T$ is in the row space of A.