1 (a) The leading variables are x_2, x_3, and x_5, while the free variables are x_1 and x_4.

(b) The equations for the homogeneous system are: $x_2 = -2x_4$, $x_3 = -3x_4$, $x_5 = 0$, so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ -2x_4 \\ -3x_4 \\ x_4 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -2 \\ -3 \\ 1 \\ 0 \end{bmatrix}.$$

A basis for the null space is thus:

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -3 \\ 1 \\ 0 \end{bmatrix} \right\}}.$$

2 A basis for the row space is

$$\{ [0,1,0,2,0], [0,0,1,3,0], [0,0,0,0,1] \}$$

3 $$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 & 2 & 0 \\ 0 & -\lambda & 1 & 3 & 0 \\ 0 & 0 & -\lambda & 0 & 1 \\ 0 & 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & -\lambda \end{vmatrix} = (-\lambda)^5 = 0 \Rightarrow \lambda = 0 \text{ is an eigenvalue}$$

of multiplicity 5. ~~The eigenvectors are simply the~~ A basis for the eigenspace is thus simply a basis for the null space of A, namely

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -3 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Thus, the set of all eigenvectors does not span $\mathbb{R}^5$.

4 $$|A-\lambda I| = \begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} -1 & 0 \\ -1 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)\left[(2-\lambda)^2 - 1\right] - (2-\lambda)$$

$$= (2-\lambda)\left[(2-\lambda)^2 - 2\right] = (2-\lambda)(\lambda^2-4\lambda+2) = 0$$

Thus $\lambda = 2$, $\lambda = 2+\sqrt{2}$, and $\lambda = 2-\sqrt{2}$ are eigenvalues.

Corresponding to $\boxed{\lambda = 2}$ we have:

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ giving } v_2 = 0,\ v_1 = -v_3,$$

so an eigenvector is $\boxed{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}$.