(1) $T(x) = \begin{bmatrix} 1 & -1 & 3 & -3 \\ 2 & 0 & 2 & 2 \end{bmatrix} x$

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(2) (a) $\begin{vmatrix} 0 & x & 1 & 0 \\ 1 & 0 & x & 1 \\ 0 & 0 & 1 & x \\ x & 0 & 0 & 1 \end{vmatrix} = -1 \begin{vmatrix} x & 1 & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{vmatrix} - x \begin{vmatrix} x & 1 & 0 \\ 0 & x & 1 \\ 0 & 1 & x \end{vmatrix}$

$= -x - x(x) \begin{vmatrix} x & 1 \\ 1 & x \end{vmatrix} = -x - x^2(x^2-1) = -x - x^4 + x^2$

$= \boxed{-x^4 + x^2 - x}$

(b) $\begin{vmatrix} x & 1 & 0 \\ 1 & x & 1 \\ 0 & 1 & x \end{vmatrix} = x \begin{vmatrix} x & 1 \\ 1 & x \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 \\ 1 & x \end{vmatrix} = x(x^2-1) - x$

$= \boxed{x^3 - 2x}$

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(3) We first normalize a: $\|a\| = \sqrt{3^2+4^2} = 5$,
so $v = \cancel{a}\frac{1}{\|a\|} a = \left(\frac{3}{5}, 0, \frac{-4}{5}\right)$, so the norm of the projection is $|u \circ v| = \left|\frac{3}{5} - \frac{4}{5}\right| = \boxed{+\frac{1}{5}}$

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(4) The projection is $(u \circ v)v$, where $v = \frac{1}{\|a\|}a$,
namely $-\frac{1}{5}\left(\frac{3}{5}, 0, \frac{-4}{5}\right) = \boxed{\left(\frac{-3}{25}, 0, \frac{4}{25}\right)}$.

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(5) There is a single multiple eigenvalue $\boxed{\lambda = 1}$
To find the eigenvectors, we have:

$[A - I] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, with R.R.E.F: $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The only eigenvector ~~is~~ corresponds to the only free variable x_1 ($x_2 = x_3 = x_4 = 0$). We have

$\boxed{v = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$

(a) Yes; $\lambda = 1$ is a multiple eigenvalue.

(b) No, one would need 4 linearly independent eigenvectors to span $\mathbb{R}^4$.

6 (a) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 1 & 2 & 3 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$

$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$ ~~The ba~~

(b) The leading variables are x_1 and x_4, and the free variables are x_2 and x_3.

(c) A basis for the row space is:
$\{[1,2,3,0], [0,0,0,1]\}.$

(d) The dimension of the row space is $\boxed{2}$.

(e) We have $x_1 = -2x_2 - 3x_3$; $x_4 = 0$.

This gives $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$

A basis for the null space is thus.

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(f) The dimension of the null space is $\boxed{2}$