Iterative Methods for Ill-Posed Problems

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Outline:

- Inverse and ill-posed problems
- Tikhonov regularization
- Regularization by truncated iteration
- Multigrid methods
- Application to image restoration

Inverse problems

- Inverse problems arise when one seeks to determine the cause of an observed effect.
  - Helioseismology: Determine the structure of the sun by measurements from earth or space.
  - Medical imaging, e.g., electrocardiographic imaging, computerized tomography.
  - Image restoration: Determine the unavailable exact image from an available contaminated version.
- Inverse problems often are ill-posed.

Ill-posed problems

A problem is said to be ill-posed if it has at least one of the properties:

- the problem does not have a solution,
- the problem does not have a unique solution,
- the solution does not depend continuously on the data.

Linear discrete ill-posed problems

$$Ax = b$$

arise from the discretization of linear ill-posed problems (Fredholm integral equations of the first kind) or, naturally, in discrete form (image restoration).

- The matrix A is of ill-determined rank, possibly singular. System may be inconsistent.
- The right-hand side b represents available data that generally is contaminated by an error.

Available contaminated, possibly inconsistents, linear system

$$Ax = b \qquad (1)$$

Unavailable associated consistent linear system with error-free right-hand side

$$Ax = \hat{b} \qquad (2)$$

Let $\hat{x}$ denote the desired solution of (2), e.g., the minimal-norm solution.

Task: Determine an approximate solution of (1) that is a good approximation of $\hat{x}$.

Define the error

$$e = \hat{b} - b \qquad \text{noise}$$

and let

$$\delta = \|e\|$$

How much damage can a little noise really do?

How much noise requires the use of special techniques?

Example: Fredholm integral equation of the 1st kind

$$\int_0^{\pi} \exp(-st)x(t)dt = 2\frac{\sinh(s)}{s}, \quad 0 \le s \le \frac{\pi}{2}.$$

Determine solution $x(t) = \sin(t)$.

Discretize integral by Galerkin method using piecewise constant functions. Code baart from Regularization Tools.

This gives a linear system of equations

$$Ax = \hat{b}, \qquad A \in \mathrm{R}^{200\times 200}, \;\; \hat{b} \in \mathrm{R}^{200}.$$

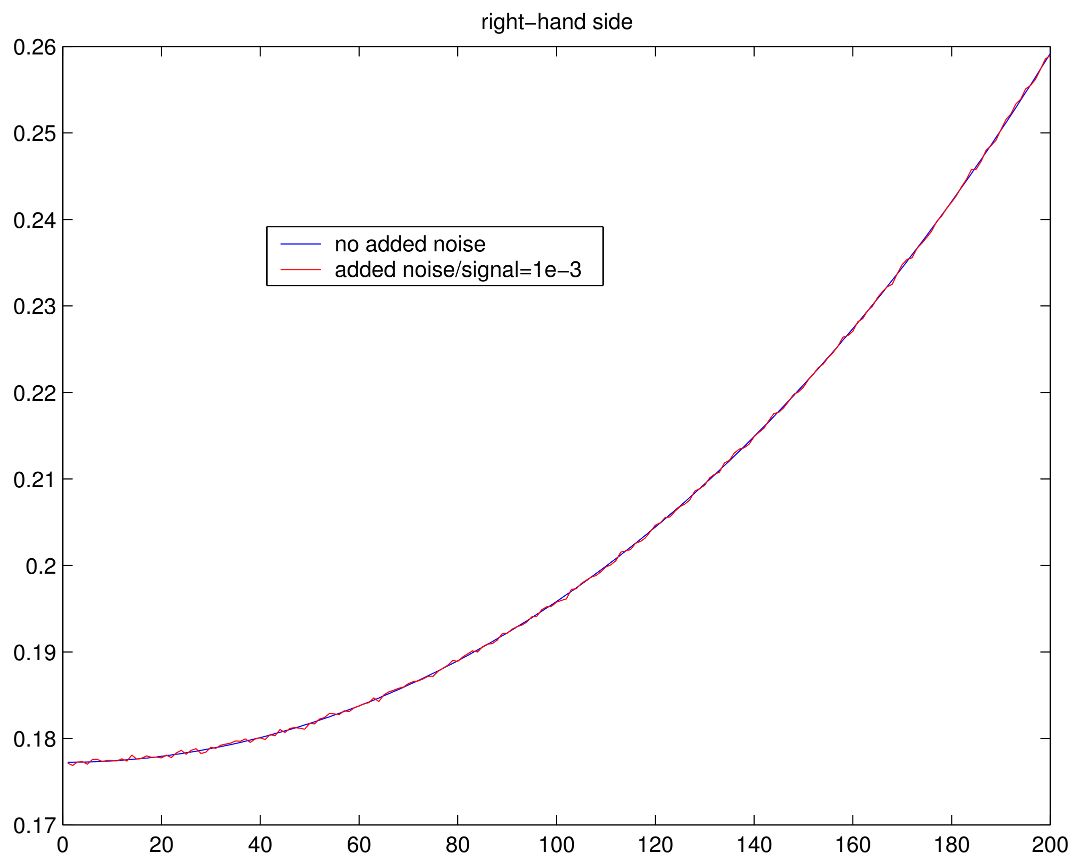
A is numerically singular.

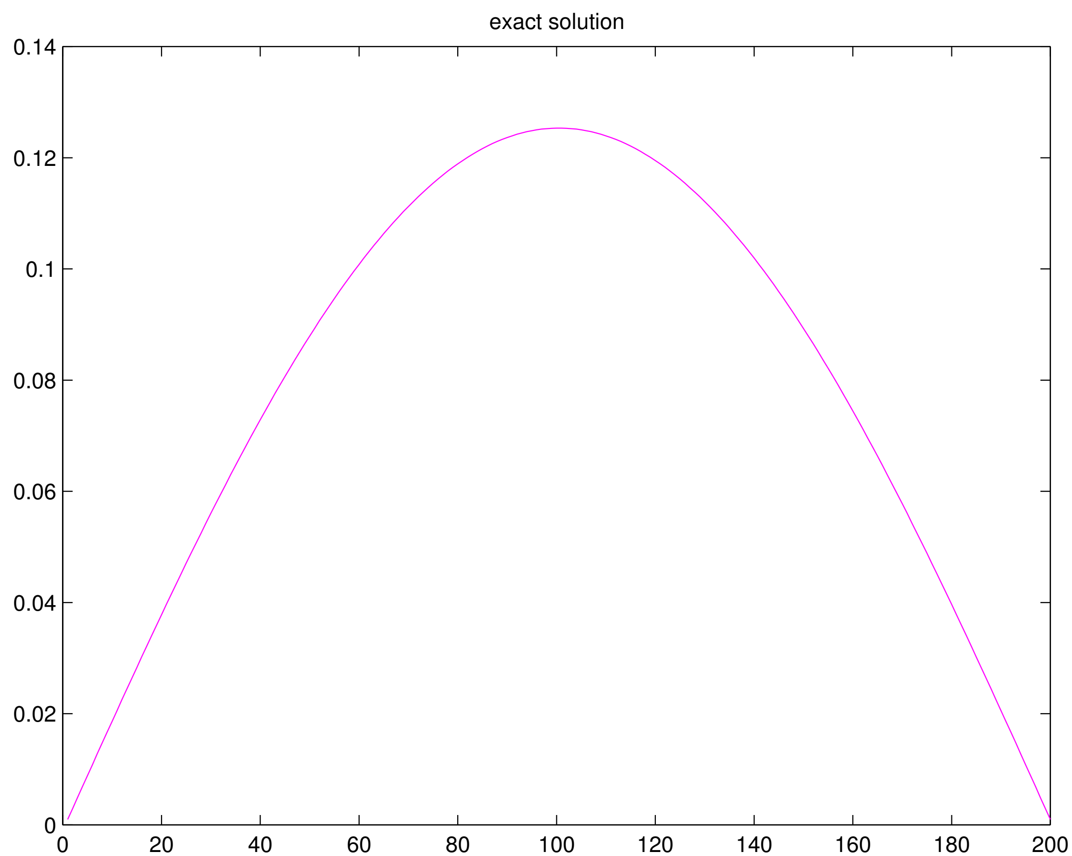
Let the “noise” vector e in b have normally distributed entries with mean zero and

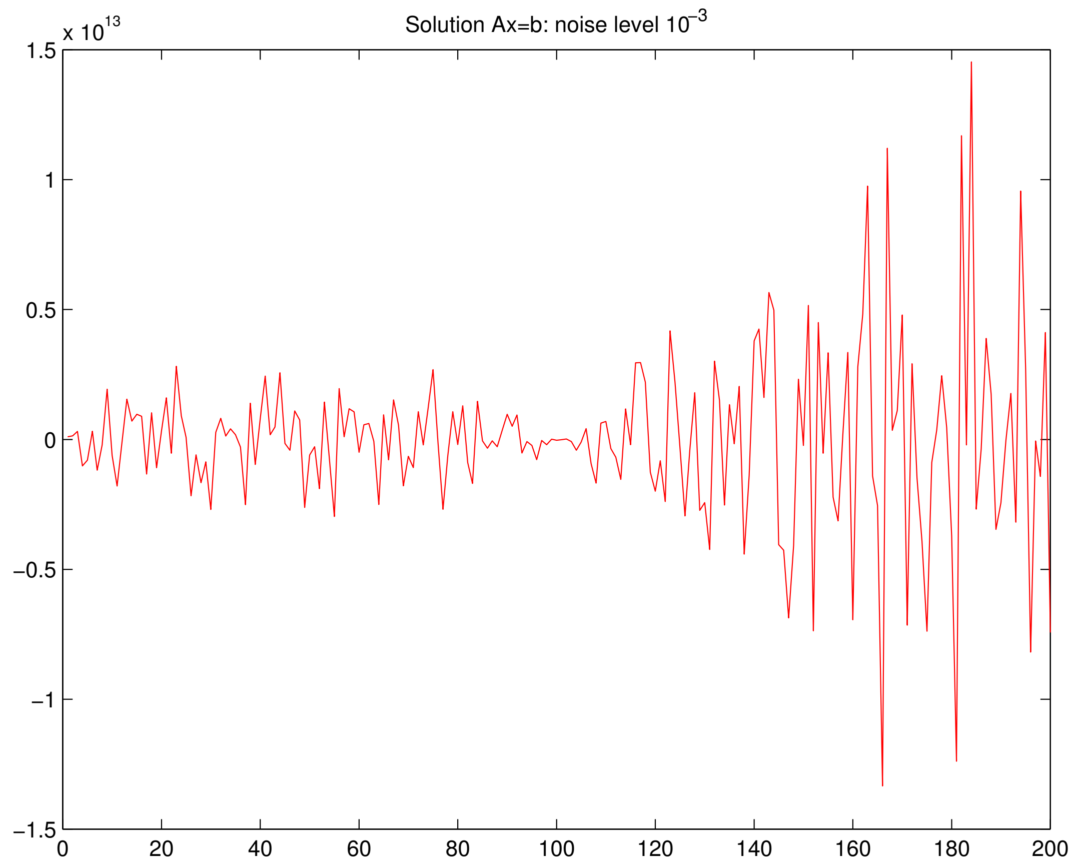
$$\delta = \|e\| = 10^{-3}\|b\|$$

$$b = \hat{b} + e$$

i.e., 0.1% relative noise


right−hand side
no added noise
added noise/signal=1e−3
0.26
0.25
0.24
0.23
0.22
0.21
0.2
0.19
0.18
0.17
0
20
40
60
80
100
120
140
160
180
200


exact solution
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0
20
40
60
80
100
120
140
160
180
200



Solution Ax=b: noise level 10−3
x 10^13
1.5
1
0.5
0
−0.5
−1
−1.5
0
20
40
60
80
100
120
140
160
180
200

Tikhonov regularization

Solve the Tikhonov minimization problem

$$\min_x \{\|Ax - b\|^2 + \mu \|Lx\|^2\},$$

where

- $A \in \mathbf{R}^{m \times n}$;
- $L \in \mathbf{R}^{p \times n}$, $p \leq n$, is the regularization operator. Common choices: $L = I$ or a finite difference operator;
- $\mu > 0$ is the regularization parameter to be determined.

Normal equations associated with Tikhonov minimization problem:

$$(A^T A + \mu L^T L)x = A^T b.$$

Have unique solution iff

$$\mathcal{N}(A) \cap \mathcal{N}(L) = \{0\}.$$

Important to determine a suitable value of μ:

$L = I$: Solution

$$x_\mu := (A^T A + \mu I)^{-1} A^T b.$$

$$\lim_{\mu \searrow 0} x_\mu = A^\dagger b, \qquad \lim_{\mu \to \infty} x_\mu = 0.$$

Regularization by truncated iteration

CGNR: CG applied to $A^*Ax = A^*b$

Define the Krylov subspace

$$\mathcal{K}_k(A^*A, A^*b) = \text{span}\{A^*b, (A^*A)A^*b, \ldots, (A^*A)^{k-2}A^*b, (A^*A)^{k-1}A^*b\}.$$

Then $x_k \in \mathcal{K}_k(A^*A, A^*b)$ and

$$\|Ax_k - b\| = \min_{x \in \mathcal{K}_k(A^*A, A^*b)} \|Ax - b\|$$

Therefore discrepancy $d_j = b - Ax_j$ satisfies

$$\|b\| \geq \|d_1\| \geq \ldots \geq \|d_k\|.$$

Stopping Criterion

Discrepancy principle

Let $\alpha > 1$ be fixed, $\|e\| = \|\hat{b} - b\| = \delta$. The iterate x_k satisfies the discrepancy principle if

$$\|Ax_k - b\| \le \alpha\delta$$

Stopping rule

Terminate the iterations as soon as iterate x_k satisfies

$$\|Ax_k - b\| \le \alpha\delta$$

$$\|Ax_{k-1} - b\| > \alpha\delta$$

Denote the termination index by k_δ.

An iterative method is a regularization method if

$$\lim_{\delta \searrow 0} \sup_{\|e\| \leq \delta} \|x_{k_\delta} - \hat{x}\| = 0$$

CGNR is a regularization method; see Nemirovskii, Hanke.

A multilevel method

Consider

$$\int_\Omega k(s,t)x(s)ds = b(t), \qquad t \in \Omega$$

Let

$$S_1 \subset S_2 \subset \ldots \subset S_k \subset L_2(\Omega) \qquad \text{nested subspaces}$$

$$R_i : L_2(\Omega) \longrightarrow S_i \qquad \text{restriction operator}$$

$$b_i = R_i b, \qquad b_i^\delta = R_i b^\delta, \qquad A_i = R_i A R_i^*$$

$$P_i : S_{i-1} \longrightarrow S_i \qquad \text{prolongation operator}$$

Cascadic multilevel method:

- Solve integral equation in S_1, map solution to S_2,
- Solve integral equation in S_2 for correction, map to S_3,
- Solve integral equation in S_3 for correction, map to S_4 ...

Assume that

$$\|b - b^{\delta}\| = \delta$$

and

$$\|b_i - b_i^{\delta}\| \leq c\delta, \quad c > 1, \quad i = 1, 2, \ldots$$

Apply CGNR or MR-II in S_1. Yields iterates $x_{1,j}$, $j = 1, 2, \ldots$. Terminate iterations as soon as

$$\|A_1 x_{1,j} - b_1^\delta\| \leq c\delta.$$

Proceed similarly on higher levels.

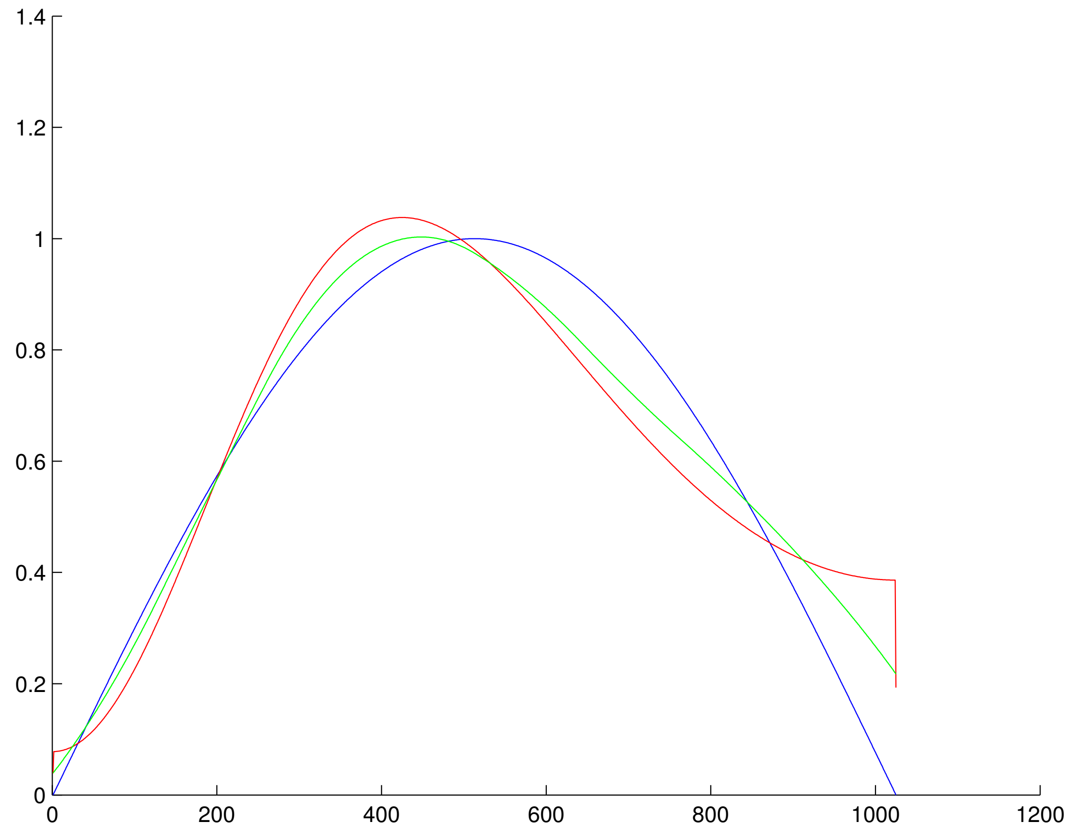
Theorem: The multilevel method outlined is a regularization method.

Example: Baart integral equation discretized by trapeziodal rule on mesh with 1025 point.

	One-Grid CGNR	
$\frac{\delta}{\|b\|}$	$m(\delta)$	$\frac{\|x^{\delta}_{8,m(\delta)}-\hat{x}\|}{\|\hat{x}\|}$
$1 \cdot 10^{-1}$	2	0.3412
$1 \cdot 10^{-2}$	3	0.1662
$1 \cdot 10^{-3}$	3	0.1657
$1 \cdot 10^{-4}$	4	0.1143

	Multilevel CGNR	
$\frac{\delta}{\|b\|}$	$m_i(\delta)$	$\frac{\|x^{\delta}_{8,m_8(\delta)}-\hat{x}\|}{\|\hat{x}\|}$
$1 \cdot 10^{-1}$	2, 1, 1, 1, 1, 1, 1, 1	0.2686
$1 \cdot 10^{-2}$	2, 2, 1, 3, 1, 1, 1, 1	0.1110
$1 \cdot 10^{-3}$	3, 3, 2, 1, 1, 1, 1, 1	0.1065
$1 \cdot 10^{-4}$	4, 3, 3, 3, 2, 1, 1, 1	0.0669

Noise level 10^{-3}: CGNR (red curve), ML-CGNR (green curve), exact solution (blue curve)



Example: Phillips integral equation discretized by trapeziodal rule on mesh with 1025 point.

	One-Grid CGNR	
$\frac{\delta}{\|b\|}$	$m(\delta)$	$\frac{\|x^{\delta}_{8,m(\delta)} - \hat{x}\|}{\|\hat{x}\|}$
$1 \cdot 10^{-1}$	3	0.0934
$1 \cdot 10^{-2}$	4	0.0248
$1 \cdot 10^{-3}$	4	0.0243
$1 \cdot 10^{-4}$	11	0.0064

	Multilevel CGNR	
$\frac{\delta}{\|b\|}$	$m_i(\delta)$	$\frac{\|x^{\delta}_{8,m_8(\delta)} - \hat{x}\|}{\|\hat{x}\|}$
$1 \cdot 10^{-1}$	2, 1, 1, 1, 1, 1, 1, 1	0.0842
$1 \cdot 10^{-2}$	5, 5, 3, 2, 1, 1, 1, 1	0.0343
$1 \cdot 10^{-3}$	8, 6, 6, 4, 3, 1, 1, 1	0.0243
$1 \cdot 10^{-4}$	9, 13, 9, 9, 5, 4, 3, 2	0.0076

Application to image restoration

Nonlinear Prolongation Operators

Return to Tikhonov regularization:

$$\min_u \left\{ \int_\Omega (h * u - f^\delta)^2 + \mu R(u) dx \right\}$$

Associated Euler-Lagrange equations:

$$\frac{\partial u}{\partial t} = -h * (h * u - f^\delta) + \mu \, D(u), \qquad u^0 = f^\delta$$

Example:

$$R(u) = |\nabla u|^2 \qquad \Longrightarrow \qquad D(u) = \Delta^2 u$$

$$R(u) = |\nabla u| \qquad \Longrightarrow \qquad D(u) = \mathrm{div}(\frac{\nabla u}{|\nabla u|}) \qquad \text{TV operator}$$

We consider a weighted total variation operator:

$$D_1(u) = |\nabla u|_\epsilon^q \, \nabla \cdot \left(\frac{\nabla u}{|\nabla u|_\epsilon^q} \right), \qquad |\nabla u|_\epsilon^q = \left(|\nabla u|^2 + \epsilon^2 \right)^{q/2}$$

with $1 \leq q \leq 2$, and a Perona-Malik operator:

$$D_2(u) = \nabla \cdot (g(|\nabla u|^2) \nabla u), \qquad g(s^2) = 1/(1 + s^2/\rho^2)$$

with g the diffusivity function.

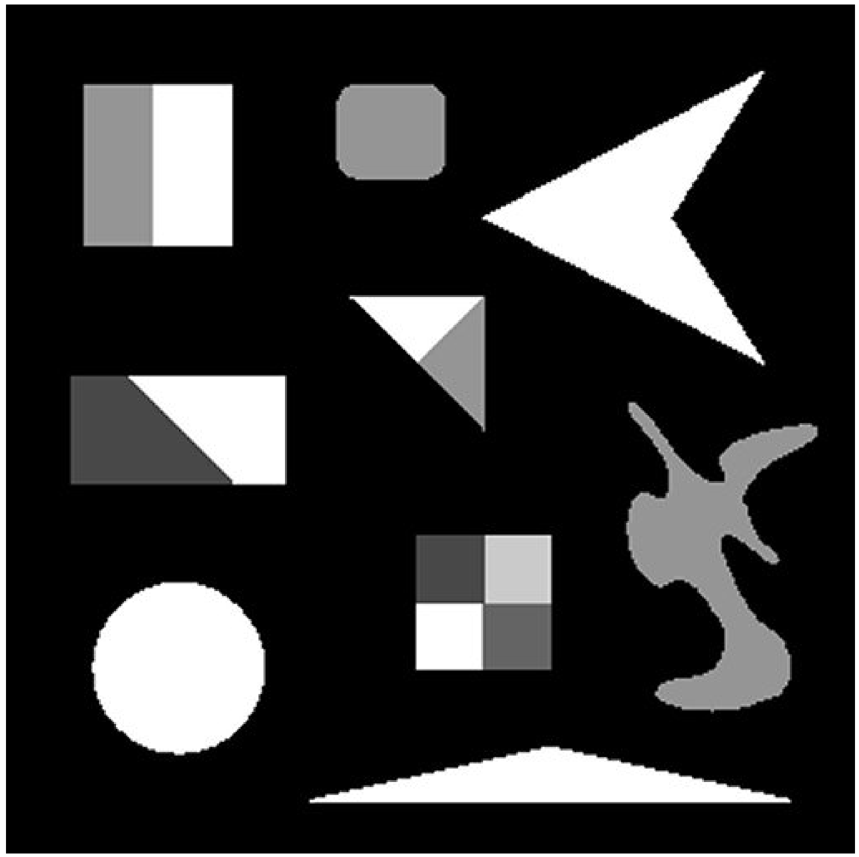
The prolongation operators are determined by integration of

$$\frac{\partial u}{\partial t} = D_j(u)$$

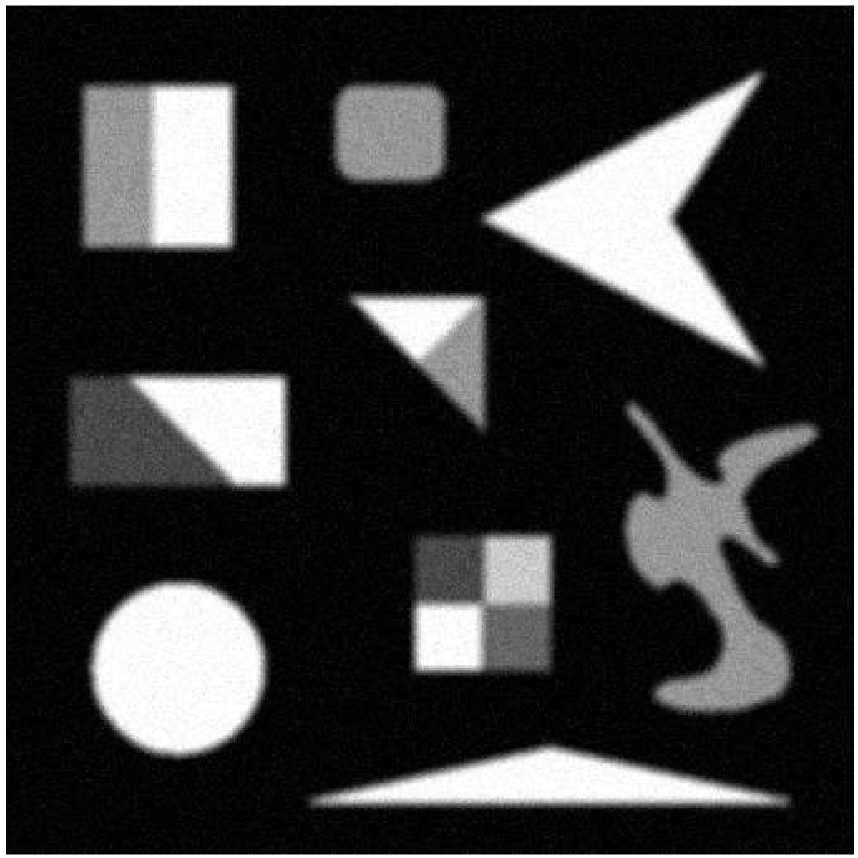
for $j = 1$ or $j = 2$.

Example: Restoration 512×512-pixel image contaminated by 10% noise and Gaussian blur.

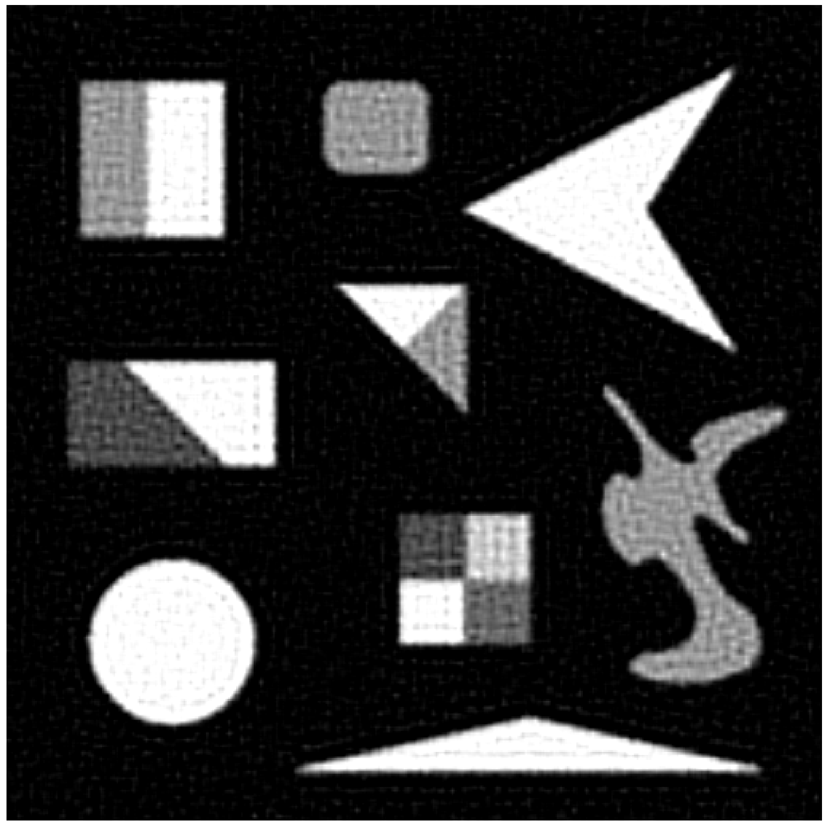
Original blur- and noise-free image



Blurred and noisy image



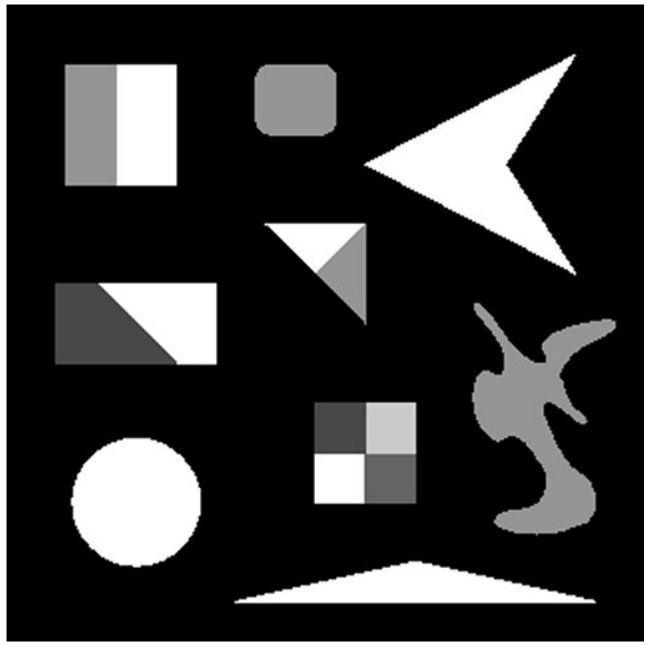
Restoration using multigrid with piecewise linear prolongation



Restoration using multigrid with D_1 prolongation (TV-norm)

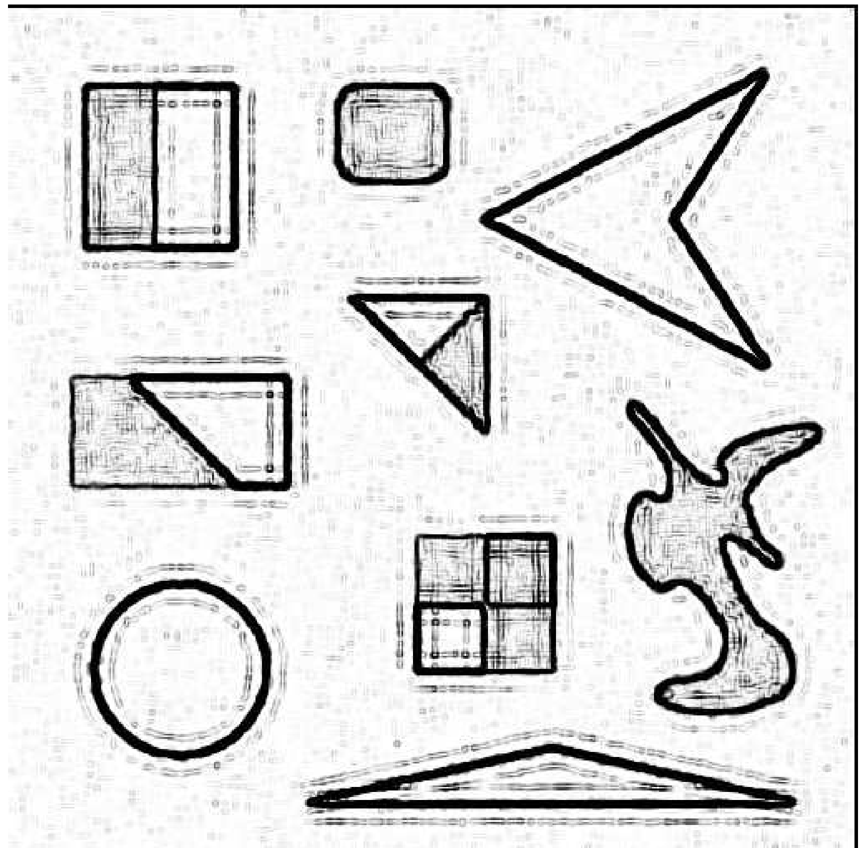
Restoration using multigrid with D_2 prolongation (Perona-Malik)

Example:

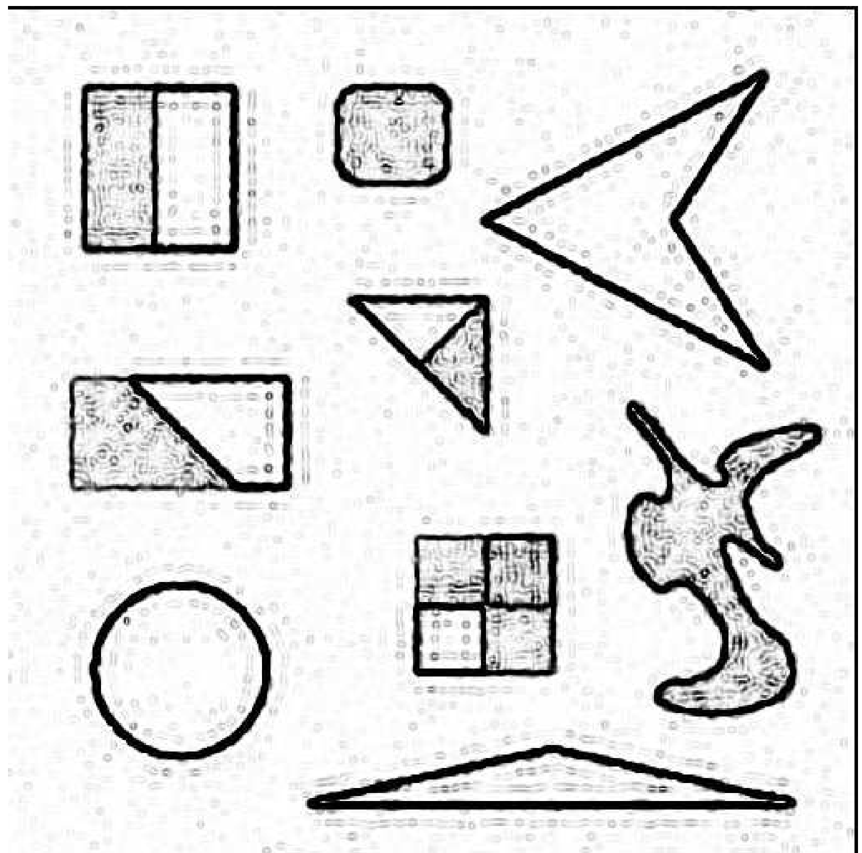


Contaminate by Gaussian blur and 0.5% noise. Restored with 1 or 3 level multigrid method. Then apply edge detector from gimp.

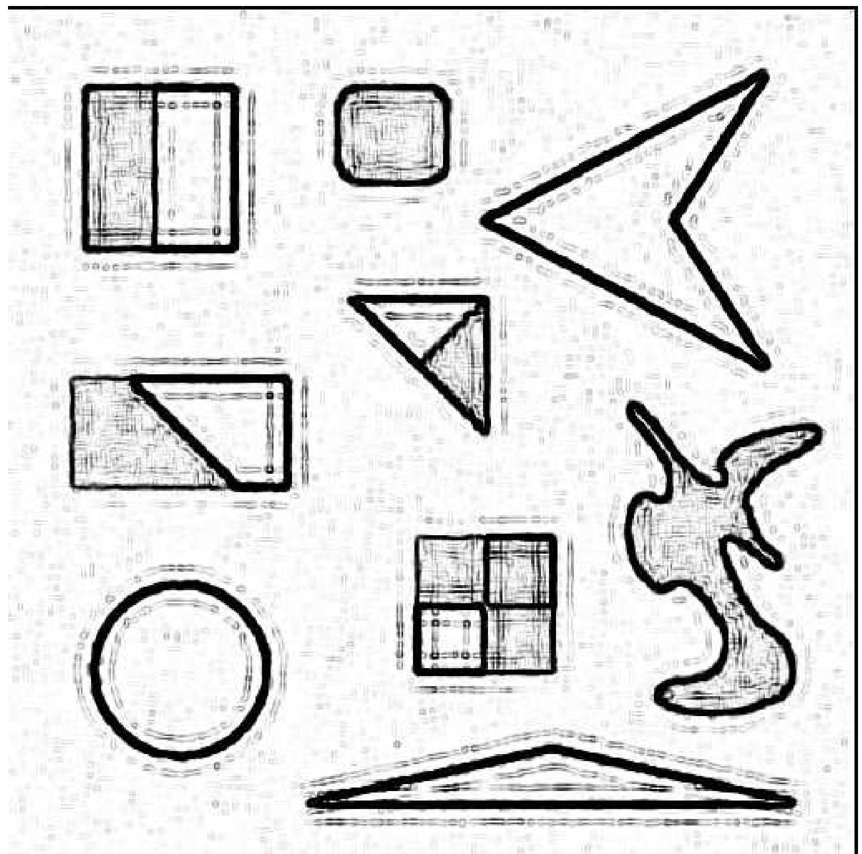
1-level CGNR



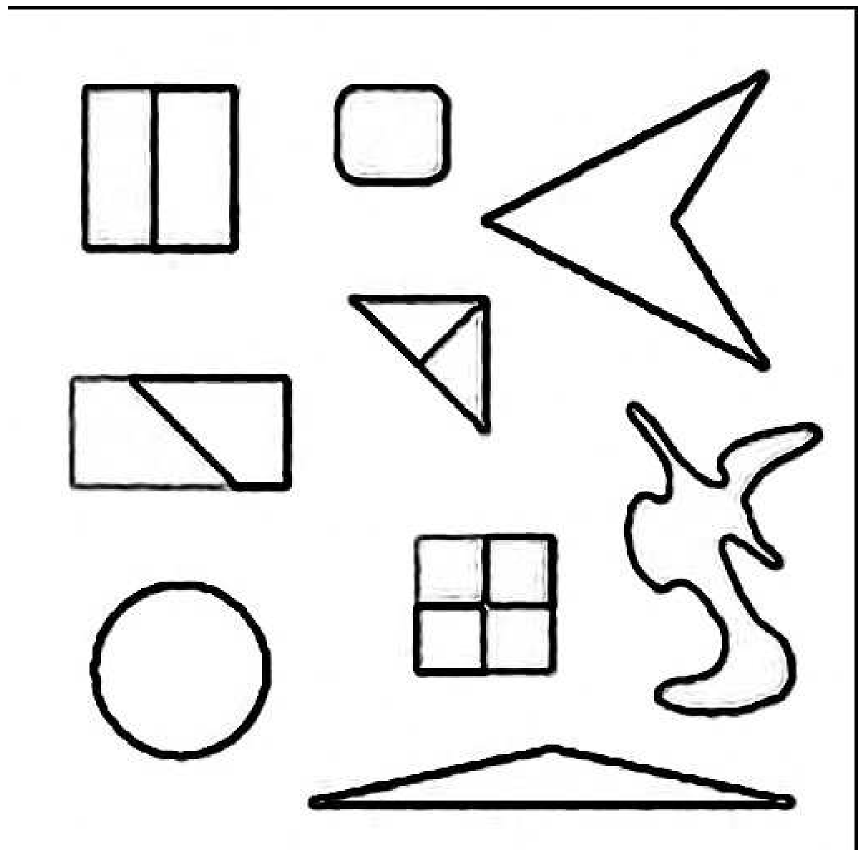
3-level multigrid with piecewise linear prolongation



1-level CGNR with D_2 applied after convergence



3-level multigrid with D_2 applied after convergence



Example: Original 256×256 image

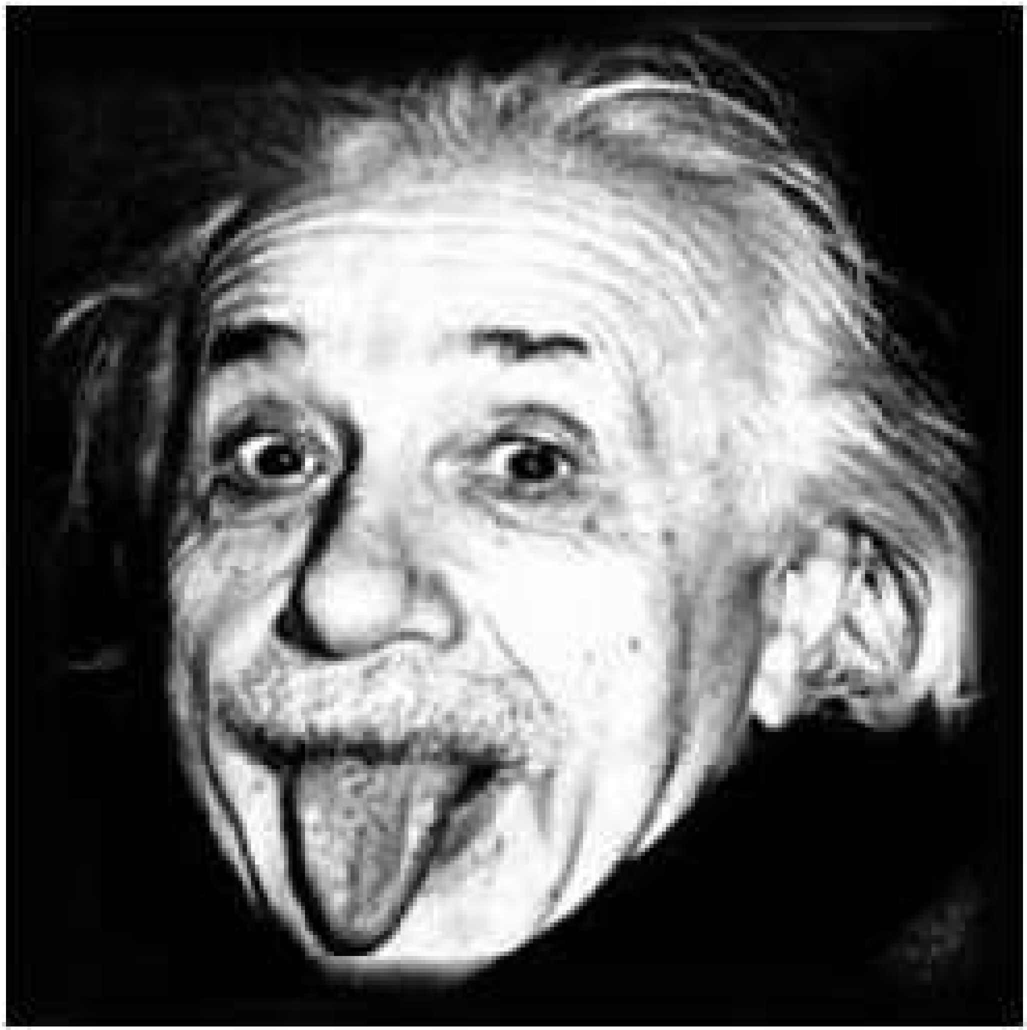
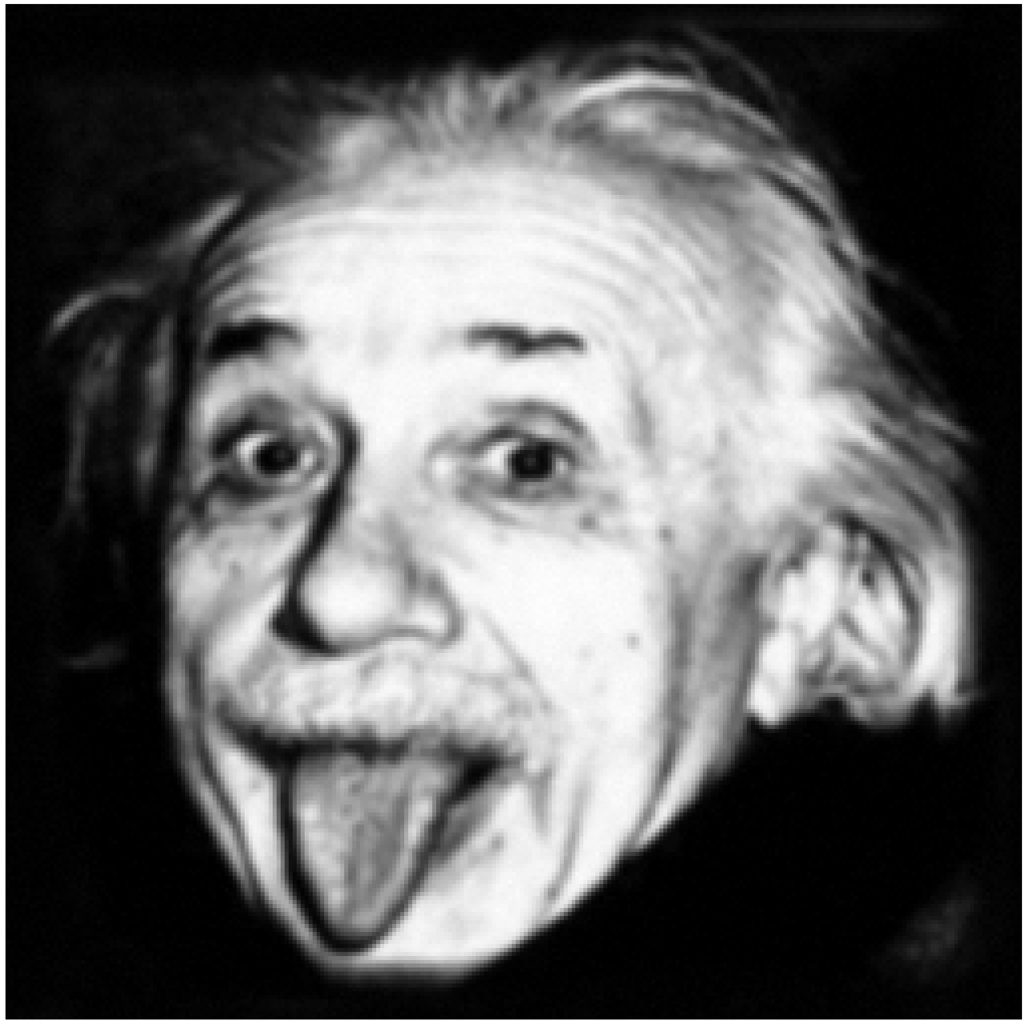
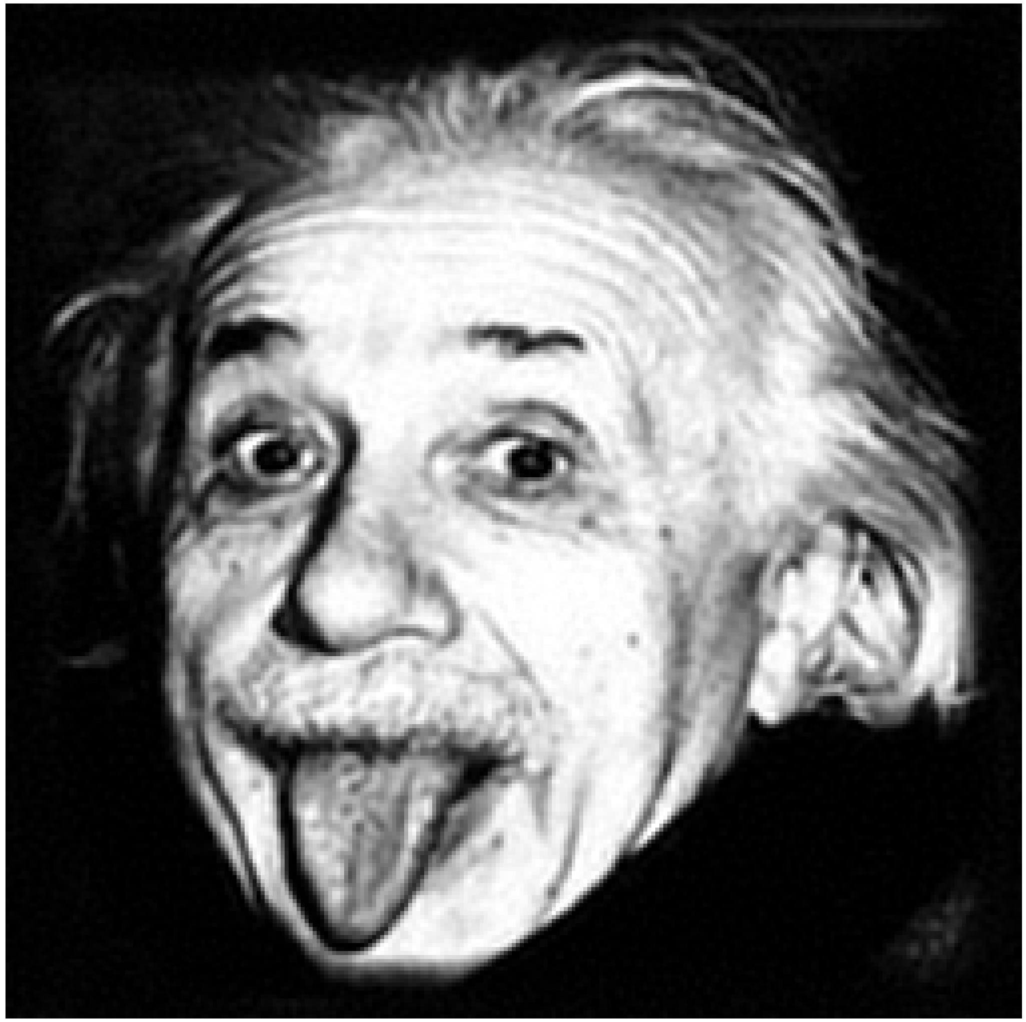


Image contaminated by Gaussian blur and 0.5% noise



Restored image using prolongation D_2



More iterations ...

Restored image using nonlinear PDE model based on D_2