A Linear Algebra Approach to Boundary Multiwavelets

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Overview

- Wavelets
- Multiwavelets
- The Boundary Problem
- The Linear Algebra Approach
  - Madych Method
  - More General Method

Refinable Functions

- A *refinable function* satisfies a recursion relation

$$\varphi(x) = \sqrt{2} \sum h_k \varphi(2x - k)$$

It is *orthogonal* if

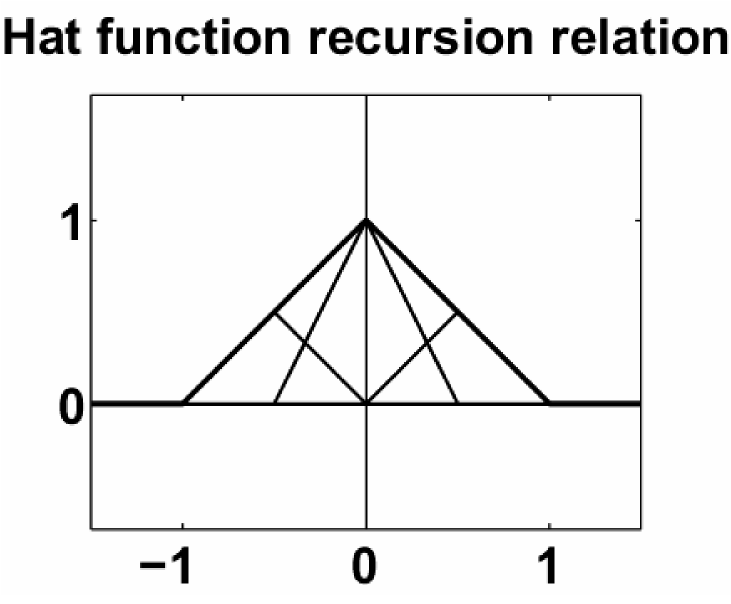
$$\langle \varphi(x), \varphi(x-k) \rangle = \delta_{0k}$$

$$\sum h_k h_{k-2l} = \delta_{0l}$$

Refinable Functions

Example: Hat Function (not orthogonal)

$$\varphi(x) = \frac{1}{2}\varphi(2x+1) + \varphi(2x) + \frac{1}{2}\varphi(2x-1)$$

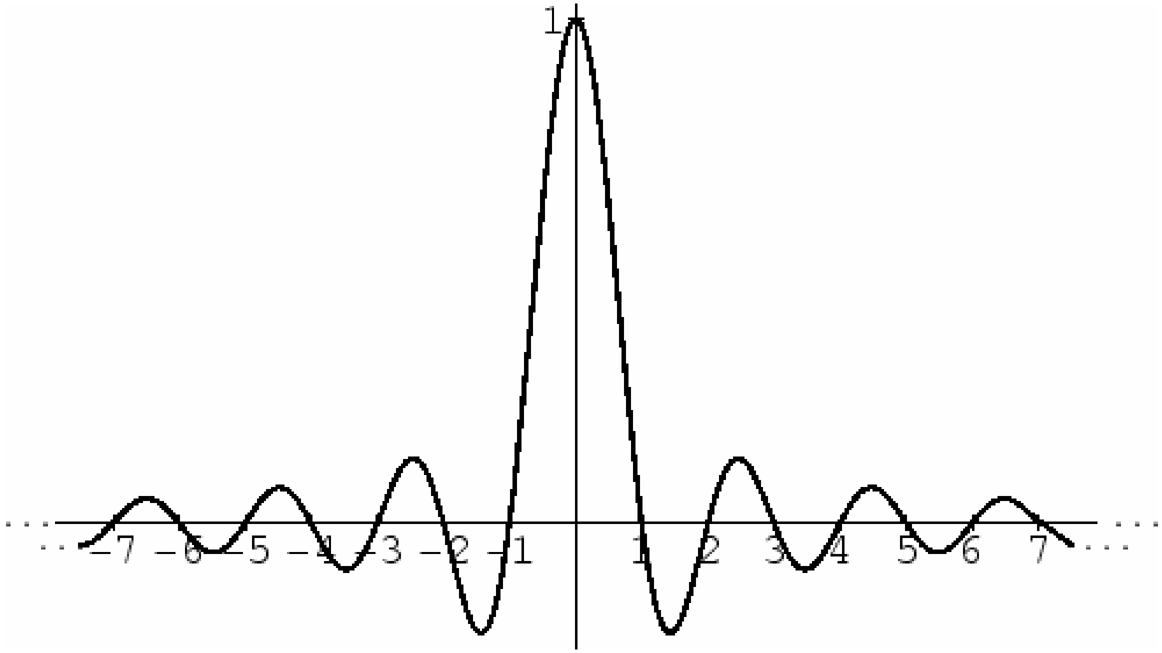


Refinable Functions

Example: sinc(x)

Not a good choice:

- not L^1
- not localized



Scaling Function

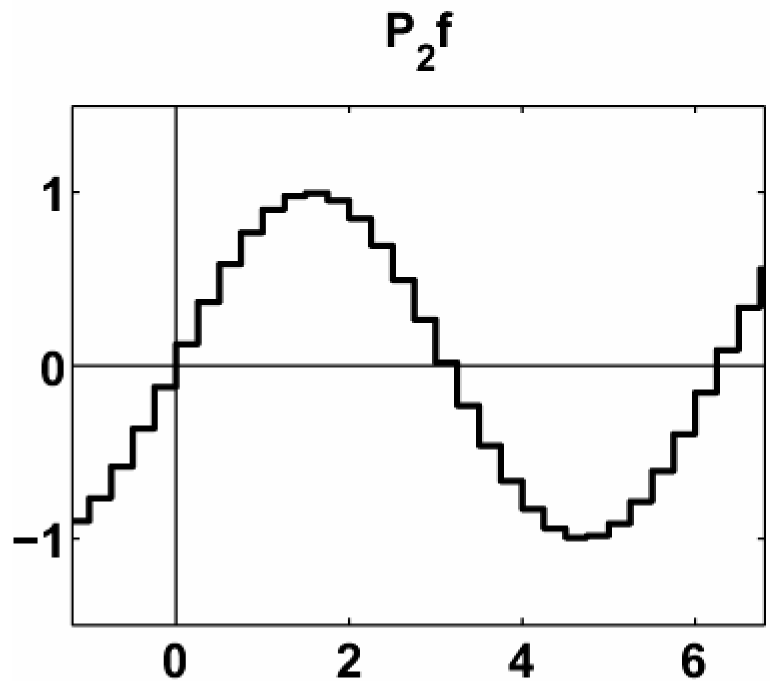
- Approximation to *f* at resolution 2^{-n}

$$\varphi_{nk}(x) = 2^{n/2}\varphi(2^n x - k)$$

$$V_n = \mathrm{span}(\varphi_{nk})$$

$$P_n f(x) = \sum \langle f, \varphi_{nk} \rangle \varphi_{nk}(x)$$

- φ is called the *scaling function*
- **Example:** Approximation to *sin* x at resolution 2^{-2} by Haar scaling function $\varphi(x) = \chi_{[0,1]}(x)$



Wavelet Function

- Fine detail in *f* at resolution 2^{-n}

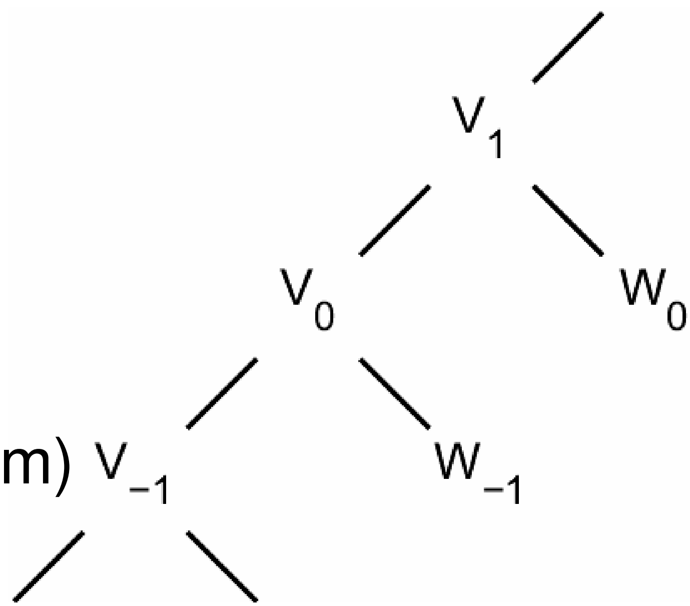
$$Q_n f(x) = P_{n+1} f(x) - P_n f(x)$$

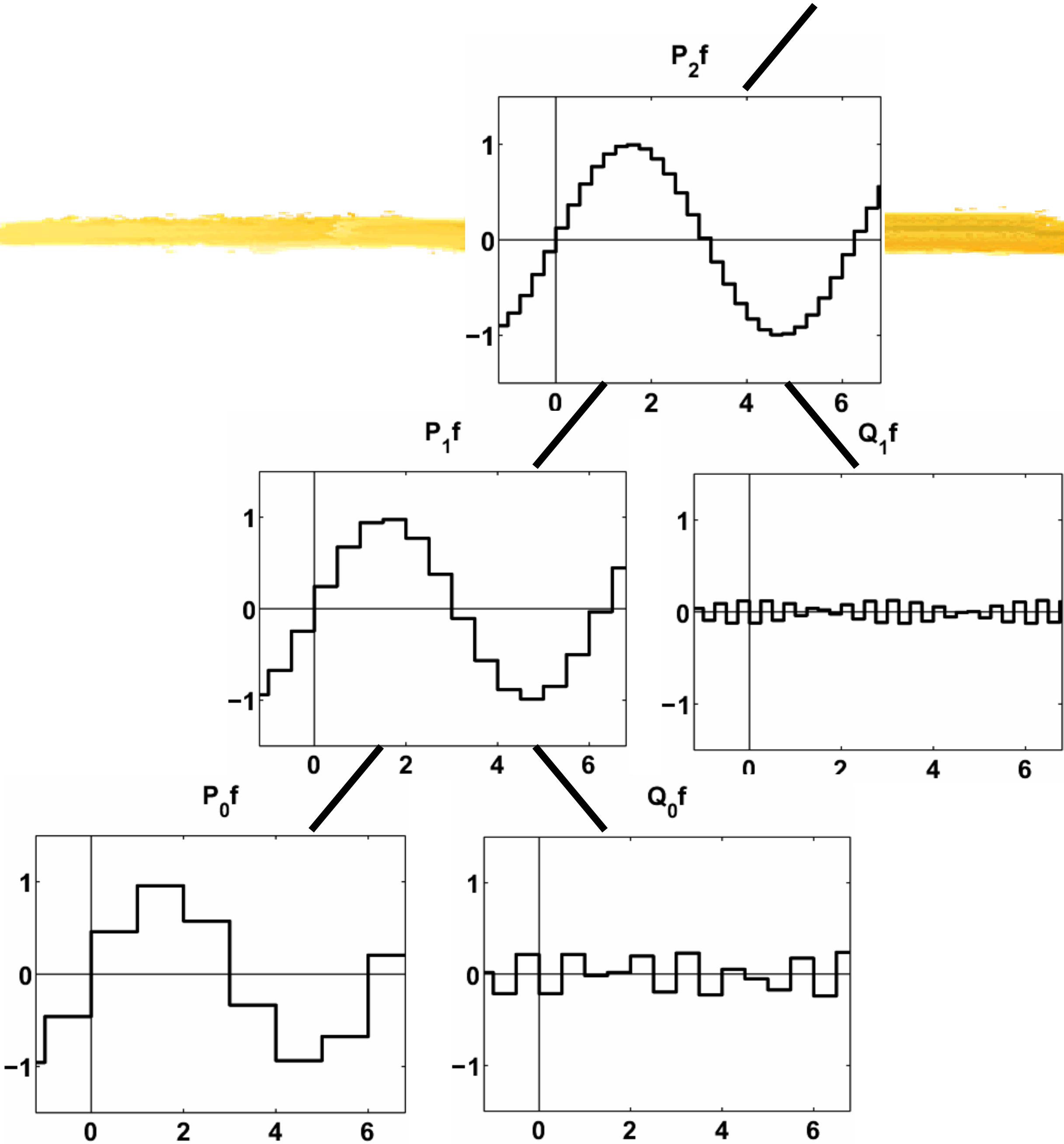
$$Q_n f(x) = \sum \langle f, \psi_{nk} \rangle \psi_{nk}(x)$$

$$W_n = \mathrm{span}(\psi_{nk})$$

- ψ is called the *wavelet function*
- DWT (Discrete Wavelet Transform)

$$P_n f = P_N f + \sum_{k=N}^{n-1} Q_k f.$$




P_2f
P_1f
Q_1f
P_0f
Q_0f

Refinable Function Vector

Refinable function vector, matrix recursion relation

$$\varphi(x) = \sqrt{2}\sum H_k \varphi(2x - k)$$

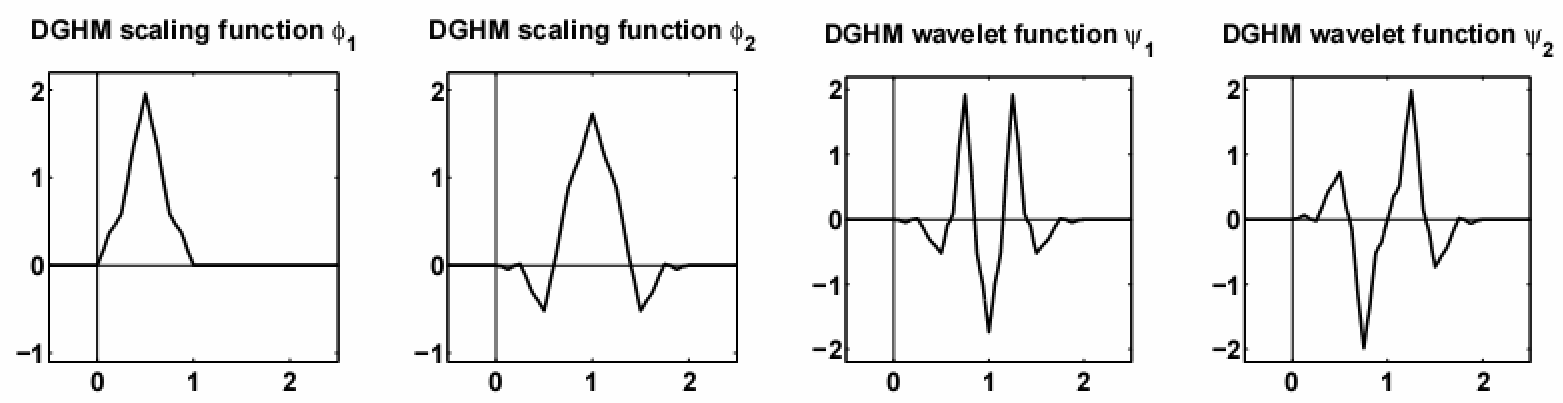
$$\varphi(x) = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \\ \vdots \\ \varphi_r(x) \end{pmatrix}$$

H_k is r x r matrix.

Multiwavelets

Example: DGHM multiwavelet

$$\phi(x) = \frac{1}{20}\left[\begin{pmatrix} 12 & 16\sqrt{2} \\ -\sqrt{2} & -6 \end{pmatrix}\phi(2x) + \begin{pmatrix} 12 & 0 \\ 9\sqrt{2} & 20 \end{pmatrix}\phi(2x-1) + \begin{pmatrix} 0 & 0 \\ 9\sqrt{2} & -6 \end{pmatrix}\phi(2x-2) + \begin{pmatrix} 0 & 0 \\ -\sqrt{2} & 0 \end{pmatrix}\phi(2x-3)\right].$$



Multiwavelets

Why are multiwavelets interesting?

- High approximation order / high smoothness combined with short support
- They can be both symmetric and orthogonal

Disadvantages:

- More complicate theory
- Need for pre/postprocessing

Discrete Multiwavelet Transform

- Start with

$$P_n s(x) = \sum_k \langle s, \phi_{n,k} \rangle \phi_{n,k}(x) = \sum_k \mathbf{s}^*_{n,k} \phi_{n,k}(x).$$

- DMWT – Direct formulation

$$\mathbf{s}_{n-1,j} = \sum_k H_{k-2j} \mathbf{s}_{n,k},$$

$$\mathbf{d}_{n-1,j} = \sum_k G_{k-2j} \mathbf{s}_{n,k}.$$

$$\mathbf{s}_{n,k} = \sum_j H^*_{k-2j} \mathbf{s}_{n-1,j} + \sum_j G^*_{k-2j} \mathbf{d}_{n-1,j}$$

Discrete Multiwavelet Transform

- DWT – Matrix formulation

$$(\mathrm{sd})_{n-1,j} = \begin{pmatrix} \mathbf{s}_{n-1,j} \\ \mathbf{d}_{n-1,j} \end{pmatrix}, \qquad T_k = \begin{pmatrix} H_{2k} & H_{2k+1} \\ G_{2k} & G_{2k+1} \end{pmatrix}.$$

$$\begin{pmatrix} \vdots \\ (\mathrm{sd})_{n-1,-1} \\ (\mathrm{sd})_{n-1,0} \\ (\mathrm{sd})_{n-1,1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \cdots & \cdots & \cdots & & & & \\ \cdots & T_{-1} & T_0 & T_1 & \cdots & & \\ & \cdots & T_{-1} & T_0 & T_1 & \cdots & \\ & & \cdots & T_{-1} & T_0 & T_1 & \cdots \\ & & & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} \vdots \\ \mathbf{s}_{n,-1} \\ \mathbf{s}_{n,0} \\ \mathbf{s}_{n,1} \\ \vdots \end{pmatrix}$$

$$(\mathrm{sd})_{n-1} = T\,\mathbf{s}_n.$$

- *T* is an infinite banded block Toeplitz matrix, $T\,T^* = I$.

The Boundary Problem

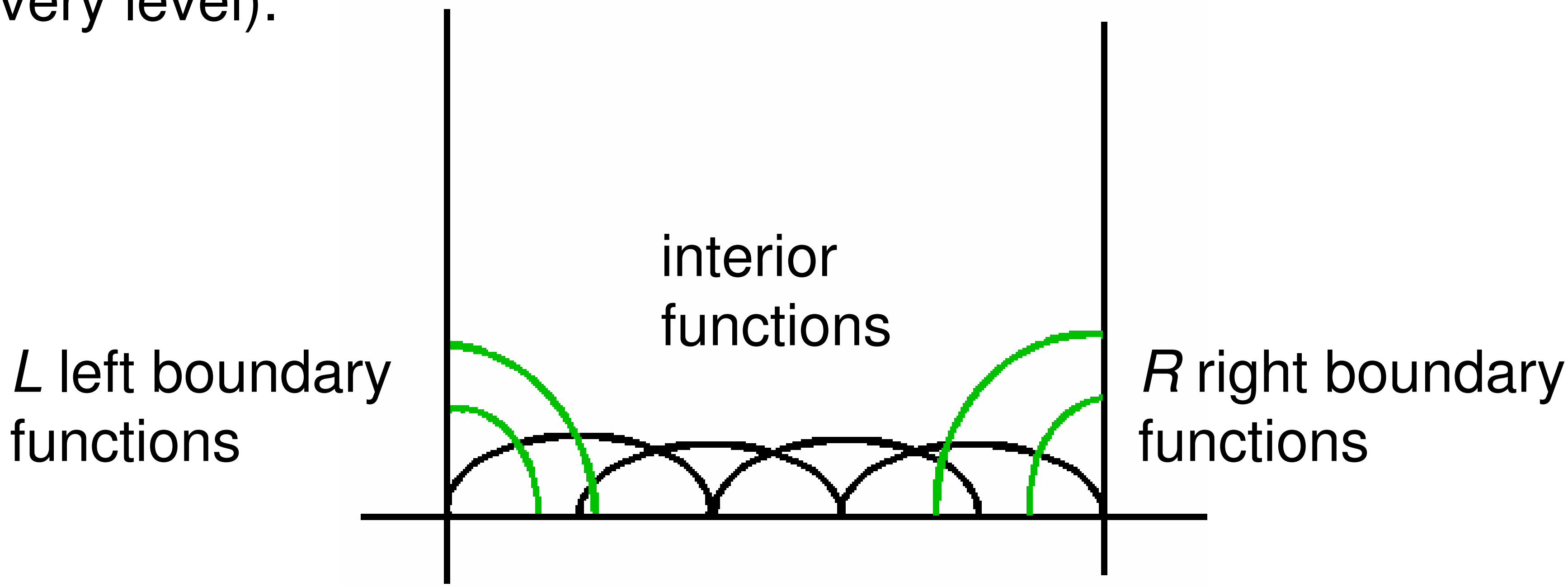
- The DMWT is naturally defined on all of R. What to do on finite interval?
  - Data Extension
    - symmetric
    - periodic
    - zero / constant / linear / ...
  - Boundary Functions
  - Linear Algebra
- One way or the other, we want to end up with

$$(\boldsymbol{sd})_{n-1} = T_n \, \boldsymbol{s}_n$$

for some finite matrix T_n with structure similar to T.

Boundary Functions

We assume we have many interior functions φ(x-k), plus some left and right boundary functions (same number at every level).



Boundary Functions

Resulting structure of T_n:

L_0 = interaction of left boundary functions with each other

L_1 = interaction of left boundary functions with inside

2L

$$\left(\begin{array}{c|cccc|c} L_0 & L_1 & & & & \\ \hline & T_0 & T_1 & & & \\ & & T_0 & T_1 & & \\ & & & \ddots & & \\ & & & T_0 & T_1 & \\ \hline & & & & R_0 & R_1 \end{array}\right)$$

2R

L

R

Linear Algebra

- For now, assume we have only two block matrices T_0, T_1

$$T_0 = \begin{pmatrix} H_0 & H_1 \\ G_0 & G_1 \end{pmatrix} \qquad T_1 = \begin{pmatrix} H_2 & H_3 \\ G_2 & G_3 \end{pmatrix}$$

- Orthogonality:

$$T_0 T_0^* + T_1 T_1^* = I$$

$$T_0 T_1^* = 0$$

Linear Algebra

- Ranks are complementary:

$$\rho_i = \operatorname{rank}(T_i)$$

$$\rho_0 + \rho_1 = 2r$$

- T_0 and T_1 have a common SVD:

$$T_0 = U \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} V^*$$

$$T_1 = U \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} V^*$$

Madych Approach

- Start with periodic case
- Use pre/postmultiplication with orthogonal matrices to convert to desired form

Illustrate with 3x3 case:

$$Q_L \begin{pmatrix} T_0 & T_1 & 0 \\ 0 & T_0 & T_1 \\ T_1 & 0 & T_0 \end{pmatrix} Q_R = \begin{pmatrix} L_0 & L_1 & 0 & 0 \\ 0 & T_0 & T_1 & 0 \\ 0 & 0 & R_0 & R_1 \end{pmatrix}$$

Madych Approach

To relate this better to what is coming next, we use SVD matrices. That is equivalent to what Madych did, even though he did not explain it this way.

$$\begin{pmatrix} U^* & 0 & 0 \\ 0 & U^* & 0 \\ 0 & 0 & U^* \end{pmatrix}\begin{pmatrix} T_0 & T_1 & 0 \\ 0 & T_0 & T_1 \\ T_1 & 0 & T_0 \end{pmatrix}\begin{pmatrix} V & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & V \end{pmatrix} = \left(\begin{array}{cc|cc|cc} I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ \hline 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ \hline 0 & 0 & 0 & 0 & I & 0 \\ 0 & I & 0 & 0 & 0 & 0 \end{array}\right)$$

Madych Approach

- Use permutation matrix on the right:

$$
\left(\begin{array}{c|c|cccc}
I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I \\
0 & 0 & 0 & 0 & I & 0 \\
0 & I & 0 & 0 & 0 & 0
\end{array}\right)
\Rightarrow
\left(\begin{array}{c|cccc|c}
I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 \\
\hline
0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 \\
\hline
0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I
\end{array}\right)
\begin{array}{l}
2r \\ \\ \\ \\ 2r
\end{array}
$$

ρ_0 ρ_1

Madych Approach

- Only requires multiplication from right
- This approach only works if $\rho_0 = \rho_1 = r$.
  - For $r = 1$, this is automatic
    (which is the only case Madych considered)
  - If $\rho_0 = \rho_1 = r$, this still works for multiwavelets
  - In general, this condition is not satisfied
  - Counterexample: DGHM multiwavelet

Different Approach

Instead, do different rearrangement

$$
\begin{pmatrix}
I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I \\
0 & 0 & 0 & 0 & I & 0 \\
0 & I & 0 & 0 & 0 & 0
\end{pmatrix}
\Rightarrow
\left(\begin{array}{c|cccc|c}
I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 \\
\hline
0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 \\
\hline
0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I
\end{array}\right)
$$

$2\rho_1$

$2\rho_0$

ρ_1 ρ_0

Longer Wavelets

What if there are more than two T_j?

$$\left(\begin{array}{ccc|ccc} T_0 & T_1 & T_2 & T_3 & 0 & 0 \\ 0 & T_0 & T_1 & T_2 & T_3 & 0 \\ 0 & 0 & T_0 & T_1 & T_2 & T_3 \end{array}\right)$$

new block T_0 new block T_1

Future Directions

- Show rank of block $T_j = n\rho_j$
- Show uniqueness of splitting
- Reduce necessary size of end-point blocks
- Show Toeplitz structure can be preserved
- Work out some examples
- Combine with pre/postprocessing