SINC–PACK, and Separation of Variables

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Abstract

This talk consists of a proof of part of Stenger's **SINC-PACK** computer package (an approx. 400-page tutorial + about 250 *Matlab* programs) that one can always achieve separation of variables when solving linear elliptic, parabolic, and hyperbolic PDE (partial differential equations) via use of *Sinc methods*.

Some examples illustrating computer solutions via **SINC-PACK** will nevertheless be given in the talk. In one dimension, **SINC-Pack** enables the following, over finite, semi–infinite, infinite intervals or arcs: interpolation, differentiation, definite and indefinite integration, definite and indefinite convolution, Hilbert and Cauchy transforms, inversion of Laplace transforms, solution of ordinary differential equation initial value problems, and solution of convolution-type integral equations. The methods of the package are especially effective for problems with (known or unknown - type) singularities, for problems over infinite regions, and for PDE problems.

In more than one dimension, the package enables solution of linear and nonlinear elliptic, hyperbolic, and parabolic partial differential equations, as well as integral equations and conformal map problems, in relatively short programs that use the above one-dimensional methods. The regions for these problems can be curvilinear, finite, or infinite. Solutions are uniformly accurate, and the rates of convergence of the approximations of **SINC-PACK** are exponential.

In Vol 1. of their 1953 text, Morse and Feshbach prove for the case of 3-dimensional Poisson and Helmholtz PDE that separation of variables is possible for essentially 13 different types of coordinate systems. A few of these (rectangular, cylindrical, spherical) are taught in our undergraduate engineering-math courses. We prove in the talk that one can ALWAYS achieve separation of variables via use of Sinc-Pack, under the assumption that calculus is used to model the PDE.

I. ONE DIMENSIONAL SINC FORMULAS

Formulas of Sinc-Pack are one dimensional. But because of *separation of variables*, these one dimensional formulas can be used to solve multidimensional PDE problems, i.e., without use of large matrices.

Sinc-Pack is package of *one dimensional approxima-*

tions, for approximating every operation of calculus. For example, let

$$F = O\, f$$

denotes a *calculus operation*, with "O" operating on a function $f = f(x)$ of one variable, x.

The corresponding *Sinc-Pack operation* takes the form

$$\mathbf{F} = \mathcal{O}\, f\,,$$

with $\mathcal{O}$ of the form

$$\mathcal{O} = \mathbf{w}\,\mathbf{O}\,V\,.$$

Here $\mathbf{w}$ is a row vector of $m = M + N + 1$ *Sinc basis functions*, $\mathbf{O}$ is an $m \times m$ matrix constructed by *Sinc-Pack* via use of the calculus operation O and the basis $\mathbf{w}$, and V is a vector operation, which transforms a function f into a vector $\mathbf{f}$, by evaluation of f at the Sinc points, z_k , i.e.,

$$\mathbf{w}(x) = (w_{-M}(x)\,,\ \ldots\,,\ w_N(x))$$

$$V\, f = (f(z_{-M}\,,\ \ldots\,,\ f(z_N))^T\,.$$

The components F_k of the resulting vector $\mathbf{F}$ satisfy the relation

$$F_k \approx F(z_k)$$

with *error* of the order of $\exp(-c\, m^{1/2})$, provided that $M \sim c\, N$, *f is analytic on the open interval (or contour)* Γ on which the operation is defined, *regardless of singularities at end-points of* Γ, and provided that F is defined on Γ. Moreover, Γ *can be a semi-infinite, or infinite interval, or even a contour in the complex plane.*

The **Sinc basis w** is interpolating, and satisfies the relation

$$\|f - \mathbf{w}\,\mathbf{f}\| = O\left(\exp\left(-c\, m^{1/2}\right)\right) \,,$$

provided that f is (*Lip alpha plus analytic*) analytic of on Γ and also, $f \in \mathbf{Lip}_\alpha(\overline{\Gamma})$.

(Optimal Convergence Rate: Burchard-Höllig, 1985) The norm $\|\cdot\|$ is the uniform norm on Γ.

Motivational Remarks.

Martensen appears to have been the first to note that if f is analytic and bounded in the strip

$$D_d = \{z \in \mathbb{C} : |\Im z| < d\} \,,$$

with d a positive number, and if f is integrable over the real line $\mathbb{R}$, then the error

$$\int_{\mathbb{R}} f(w)\, dw - h \sum_{k \in \mathbb{Z}} f(k\, h)$$

is of the order of $\exp(-2\,\pi\, d/h)$. It was later shown by Stenger that if $\varphi : \Gamma \to \mathbb{R}$, and if φ is also a conformal map of a domain of analyticity D of a function F onto D_d, then by setting $t = \varphi^{-1}(w)$ in the integral of $\int_\Gamma F(t)\, dt$ one gets the same order of error for the difference

$$\int_\Gamma F(t)\,dt - \sum_{k\in\mathbb{Z}} w(k)\,F(z_k)\,,$$

with $z_k = \varphi^{-1}(k\,h)$, and with $w_k = h/\varphi'(z_k)$.

This trapezoidal rule is obtainable via direct integration of the Sinc expansion of the function

$$f(t) = \sum_{k\in\mathbb{Z}} \mathrm{sinc}((t-k\,h)/h)\,f(k\,h)\,,$$

for which the uniform difference on $\mathbb{R}$ between f and this expansion is of the order of $\exp(-\pi\,d/h)$. The replacement of t by $\varphi^{-1}(t)$ in this expansion enables exponentially accurate approximations of every operation of calculus.

Sample Sinc-Pack Procedures:

(i) **Interpolation**

There exists an explicit Sinc basis $(\omega_{-N}\,,\ \ldots\,,\ \omega_N)^T$, such that, if, e.g., F us analytic on a finite interval (a,b) , and if also $f \in \mathbf{Lip}_\alpha(a,b)$, then

$$f - \sum_{k=-N}^{N} (\omega_k)\,f_k = \mathcal{O}(\exp(-c\,N^{1/2}))\,,$$

with $z_k = \varphi^{-1}(k\,h)$, with $f_k = f(z_k)$, and with c a positive constant. This type of approximation also translates to nonlinear functions, e.g., if g is analytic on the range of f, then

$$g\circ f - \sum_{k=-N}^{N} (\omega_k)\ g(f_k) = \mathcal{O}(\exp(-c\,N^{1/2}))\,,$$

and this property enables effective approximation of solutions to nonlinear differential equations.

(ii) **Quadrature**.

$$I = \int_0^\infty \frac{dx}{x^{1/2}\,(1+x)} = \pi.$$

Ans.: Use quadd1.m

(iii) **Indefinite Integration**. Use the notation, for $x \in (a, b)$,

$$\begin{aligned} G^+(x) &= (\mathcal{J}^+\, g)(x) = \int_a^x g(t)\, dt\,, \\ G^-(x) &= (\mathcal{J}^-\, g)(x) = \int_x^b g(t)\, dt\,. \end{aligned}$$

Then

$$\mathbf{G}^\pm = \mathbf{J}^\pm \mathbf{g}\,,$$

with $\mathbf{J}^\pm$ a matric of order m.

For example,

$$1 - \gamma(x, 2/3) = \frac{1}{\Gamma(2/3)} \int_x^\infty t^{-1/3}\, e^{-t}\, dt\,, \quad x \in (0, \infty)\,,$$

$$e^x - 1 = \int_0^x e^t\, dt\,.$$

Ans.: Use indef.example.m

(iv) **Indefinite Convolution**.

Model integrals over $(a, b) \subseteq \mathrm{I\!R}$:

$$\begin{aligned} p^+(x) &= \int_a^x f(x-t)\, g(t)\, dt \\ p^-(x) &= \int_x^b f(t-x)\, g(t)\, dt\,. \end{aligned}$$

Use "Laplace transform". Assume that for all $\Re s > 0$,

$$\mathcal{F}(s) = \int_0^c f(t)\, e^{-t/s}\, dt\,, \quad c \geq (b-a)\,.$$

Then (St, 1993)

$$p^{\pm} = \mathcal{F}(\mathcal{J}^{\pm})\, g$$

But, can prove, since

$$\mathcal{J}^{\pm}\, g \approx \mathbf{w}\, \mathbf{J}^{\pm}\, \mathbf{g}$$

we also get, if $p^{\pm} \in$ *Lip alpha* $+$ *analytic*

$$\mathcal{F}(\mathcal{J}^{\pm})\, g \approx \mathbf{w}\, \mathcal{F}(\mathbf{J}^{\pm})\, \mathbf{g}\,.$$

Evaluation: If $\mathbf{J}^{\pm} = X_{\pm}\, S\, X_{\pm}^{-1}$ with S a diagonal matrix, then

$$\mathcal{F}(\mathbf{J}^{\pm}) = X_{\pm}\, \mathcal{F}(S)\, X_{\pm}^{-1}\,.$$

For example,

$$\begin{aligned} F_1(x) &= \int_0^x |x-t|^{-1/2}\, t^{-1/2}\, dt\,, \\ F_2(x) &= \int_x^1 |x-t|^{-1/2}\, t^{-1/2}\, dt\,, \end{aligned}$$

whose solution are

$$
\begin{aligned}
F_1(x) &= \begin{cases} 0 & \text{if} \quad x = 0; \\ \pi & \text{if} \quad x > 0\,, \end{cases} \\
F_2(x) &= \log\left(\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}\right).
\end{aligned}
$$

See ex_3.6_abel.m .

(v) **More General Convolutions.** Several generalizations of the above are possible. For today's talk, we consider only

$$
r(x) = \int_a^x k(x-t,t)\,dt\,, \quad x \in (a,b)\,.
$$

The Sinc approximation of this integral is the "key" for achieving separation of variables when solving two dimensional linear PDE over curvilinear regions.

We need the "Laplace transform",

$$
K(s,t) = \int_0^c k(x,t)\,\exp(-x/s)\,dx\,, \quad c \geq (b-a)\,.
$$

There exist functions $f_\nu^n \in \mathbf{L}^1(0,b-a)$ and $g_\nu^n \in \mathbf{L}^1(a,b)$ such that

$$
k(\xi,t) = \lim_{n\to\infty} \sum_{\nu=1}^{n} f_\nu^n(\xi)\,g_\nu^n(t)\,,
$$

for all $(\xi,t) \in (0,b-a)\times(a,b)$. Take "Laplace transform" with respect to ξ , get

$$
K(s,t) = \lim_{n\to\infty} \sum_{\nu=1}^{n} \mathcal{F}_\nu^n(s)\,g_\nu^n(t)\,.
$$

Hence, performing m–vector Sinc convolution, we get $r(x) \approx \mathbf{r}$, with

$$\mathbf{r} = \lim_{n \to \infty} \sum_{\nu=1}^{n} \mathcal{F}_\nu^n(\mathbf{J}^+)\, \mathbf{g}_\nu^n \,.$$

where $\mathbf{w}\,\mathbf{J}^+\,\mathbf{g} \approx \mathcal{J}^+\,g$. That is, with $\mathbf{J}^+ = X\,S\,X^{-1}$, $X = [x_{ij}]$, $S = \mathrm{diag}(s_{-M}\,,\ \ldots\,,\ s_N)$, and $X^{-1} = [x^{ij}]$, and z_k denoting Sinc points, we get, for the i^{th} component of $\mathbf{r}$,

$$\begin{aligned} r_i &= \lim_{n \to \infty} \sum_{\nu=1}^{n} \sum_{\ell} x_{i\,\ell} \sum_{k} \mathcal{F}_\nu^n(s_\ell)\, x^{\ell\,k}\, g_\nu^n(z_k) \\ &= \sum_{\ell} x_{i\,\ell} \sum_{k} x^{\ell\,k} \lim_{n \to \infty} \sum_{\nu=1}^{n} \mathcal{F}_\nu^n(s_\ell)\, g_\nu^n(z_k) \\ &= \sum_{\ell} x_{i\,\ell} \sum_{k} x^{\ell\,k}\, K(s_\ell\,, z_k) \end{aligned}$$

If we now set $\mathbf{q} = (q_{-M}\,,\ \ldots\,,\ q_N)^T$, with

$$q_\ell = \sum_{k} x^{\ell\,k}\, K(s_\ell\,, z_k)\,,$$

then

$$\mathbf{r} = X\,\mathbf{q}\,.$$

II. “LAPLACE TRANSFORMS” OF GREEN’S FUNCTIONS

Sinc-Pack contains derivations of the multidimensional “Laplace transforms” of all of the known free space Green’s functions for elliptic, parabolic, and hyperbolic linear PDE. For example, the "Laplace transform" of the free space Green's function

$$\mathcal{G}(x,y) = \frac{1}{2\pi}\,\log\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

is just

$$G(s,\sigma) = \int_0^\infty \int_0^\infty \exp\left(-\frac{x}{s} - \frac{y}{\sigma}\right)\, G(x,y)\, dx\, dy$$

$$= \left(\frac{1}{s^2} + \frac{1}{\sigma^2}\right)^{-1} \cdot$$

$$\cdot \left(-\frac{1}{4} + \frac{1}{2\pi}\left(\frac{\sigma}{s}\,(\gamma - \log(\sigma)) + \frac{s}{\sigma}\,(\gamma - \log(s))\right)\right)$$

A particular solution to the PDE

$$\nabla^2 u(x,y) = -e(x,y)\,, \quad (x,y) \in Q\,, (a,b)\times(c,d)$$

with $Q = (a,b)\times(c,d)$ is given by

$$u(x,y) = \int\int_Q \mathcal{G}(x-\xi\,, y-\eta)\, e(\xi\,,\eta)\, d\xi d\eta\,.$$

The function $\mathcal{G}$ in this integral has a singularity at the interior point $(\xi\,,\eta) = (x,y)$. This singularity can be moved to the boundary by splitting the integral into 4, in the form

$$u(x,y) = (\int_a^x \int_c^y + \int_x^b \int_c^y + \int_a^x \int_y^d + \int_x^b \int_y^d) G\, e\, d\xi d\eta$$

Each of these 4 is a product integral; For example, if in the first we fix y and η, then the integral $\int_a^x \cdots$ is just a one dimensional convolution of the type p already considered. Next, fixing x and y, we again get the same type of one dimensional convolution. E.g. if $Q = (0,1) \times (0,1)$, and we seek an approximation of the form

$$u(x,y) \approx \sum_{i=-N}^{N} \sum_{j=-N}^{N} \omega_i(x)\, U_{ij}\, \omega_j(y)\,,$$

then, by selecting two indefinite integration matrices, one $A = XSXi$ $(Xi = X^{-1})$ for integration from 0 to x and another, $B = YSYi$ $(Yi = Y^{-1})$, for integration from x to 1, with $S = \mathrm{diag}(s_{-N}\,,\ \ldots\,,\ s_N)$, set $E = e(z_i\,, z_j)$, $G = [G(s_i\,, s_j)]$, then we can compute a matrix $U = [U_{ij}]$ via the following 4-line Matlab program:

$$\begin{array}{lclcl}
U & = & & & X * (G. * (Xi * E * Xi.')) * X.'; \\
U & = & U & + & Y * (G. * (Yi * E * Xi.')) * X.'; \\
U & = & U & + & X * (G. * (Xi * E * Yi.')) * Y.'; \\
U & = & U & + & Y * (G. * (Yi * E * Yi.')) * Y.';
\end{array}$$

An approximate solution particular solution at an arbitrary point $(x,y) \in Q$ is then given by

$$u(x,y) \approx \mathbf{w}(x)\, U\, (\mathbf{w}(y))^T\,.$$

This solution is, in fact, uniformly accurate on Q, even when e has singularities on the boundary of Q.

The same type of procedure works for elliptic, parabolic, and hyperbolic PDE.

III. BASIC ARCS AND REGIONS.

Definitions.

(i) *Arc*:

$$\Gamma = \{\bar{\rho} = (x, y) \in \mathbf{R}^2 : x = \xi(t)\,,\ y = \eta(t)\,,\quad 0 \le t \le 1\}\,,$$

with ξ and η analytic on $(0, 1)$.

(ii) *Curve*: A union $\mathcal{B}^1$ of n_1 arcs Γ_j, with $\bar{\rho}_j(1) = \bar{\rho}_{j+1}(0)$, $j = 1\,,\ 2\,,\ \ldots\,,\ n_1 - 1$.

A problem on a curve can thus be transformed into a system of n_1 problems on $(0, 1)$.

(iii) *Contour*: A curve, $\mathcal{B}^1$, with $\bar{\rho}_{n_1}(1) = \bar{\rho}_1(0)$.

Planar Regions. These are a union of at most n_2 rotations of regions of the form

$$\mathcal{B}^2_J = \{(x, y) : a1_J < x < b1_J,\ a2_J(x) < y < b2_J(x)\}\,,$$

$J = 1\,,\ 2\,,\ \ldots\,,\ n_2$, and with the property that $a2$ and $b2$ are arcs, and such that any two regions $\mathcal{B}^{(2)}_J$ and $\mathcal{B}^{(2)}_K$ with $K \ne J$ share at most a common arc. Such regions $\mathcal{B}^{(2)}_J$ can easily be represented as transformations of a square $Q^2 = (0, 1) \times (0, 1)$ via the transformation $(x, y) = \mathcal{T}_J(\xi\,, \eta)$ defined by

$$\begin{aligned} x &= a1_J + (b1_J - a1_J)\,\xi \\ y &= a2_J(x) + (b2_J(x) - a2_J(x))\,\eta\,. \end{aligned}$$

Note that $\partial\mathcal{B}^2$, the boundary of $\mathcal{B}^2$ consists of at most a finite number of contours.

Note also, that a problem over $\mathcal{B}^2$ can thus be transformed into a system of n_2 problems over Q^2.

IV. ANALYTICITY.

(i) We shall denote by $\mathbf{X}^1 = \mathbf{X}^1(\mathcal{B}^1)$ the family of all functions $f(x)$ that are defined on a curve $\mathcal{B}^1$, and are analytic on each arc Γ_j of $\mathcal{B}^1$. (We may add additional specifications of $\mathbf{X}^1$ depending on a particular problem, e.g., not only analyticity, but also *analyticity + Lip-alpha*, or even, analyticity + certain rates of blow-up, but such that the solution to the problem should belong to analyticity + Lip-alpha.

(ii) In two dimensions, denote by f_J the function f restricted to $\mathcal{B}^2_J$. We shall denote by $\mathbf{X}^2 = \mathbf{X}^2(\mathcal{B}^2)$ the family of all functions $f(x,y)$ that are defined and have a power series expansion, i.e.,

$$f_J(x,y) = \sum_{(j,k)\in\mathbb{Z}^2_+} \frac{f^{(j,k)}(x_1,y_1)}{j\,!\,k\,!}(x-x_1)^j\,(y-y_1)^k$$

with $\mathbb{Z}_+ = (0\,,\ 1\,,\ \ldots,$ for all $(x_1\,,y_1)$ in each region $\mathcal{B}^2_J$, and also, for all $(x_1\,,y^1)$ on the interior of each boundary arc of $\mathcal{B}^2_J$.

Theorem: *If* $f \in \mathbf{X}^2$ *, and if* $\bar\rho$ *is an arc either in a region* $\mathcal{B}^2_J$ *, or on a boundary arc of* $\mathcal{B}^2_J$ *, then* $f\circ\bar\rho$ *is an analytic function on* $(0,1)$ *.*

Now suppose that $\mathcal{P}$ is an arc in $Q^2 = (0,1)\times(0,1)$, defined by $\zeta(t) = (\xi(t)\,,\eta(t))$, $0 < t < 1$, and let $\mathcal{T}_J(\xi\,,\eta)$ be defined as above. Then $f \in \mathbf{X}^2 \longrightarrow g(t) = f\circ\mathcal{T}(\xi(t)\,,\eta(t))$ is an analytic function of t on $(0,1)$. Moreover, if f is also of class $\mathbf{Lip}_\alpha$ on each region $\mathcal{B}_J^2$, then $g \in \mathbf{Lip}_\alpha[0,1]$.

V. MOTIVATING 1-d ANALYTICITY

Let $F = F(x,y)$ be defined on $Q^2 = (0,1) \times (0,1)$, such that $F \in \mathbf{X}^2$, and let O^x and O^y denote calculus operations, and let $\mathcal{O}^x$ and $\mathcal{O}^y$ denote respectively, the corresponding Sinc approximations of these operations. These operators typically commute with one-another, and we can thus bound the error $\|O^x\, O^y\, F - \mathcal{O}^x\, \mathcal{O}^y\, F\|$ in the following manner:

$$
\begin{aligned}
&\|O^x\, O^y\, F - \mathcal{O}^x\, \mathcal{O}^y\, F\| \\
&\quad \le \|O^y\, (O^x\, F - \mathcal{O}^x\, F)\| + \|\mathcal{O}^x\, (O^y\, F - \mathcal{O}^y\, F)\|\,, \\
&\quad \le \|O^y\|\, \|O^x\, F - \mathcal{O}^x\, F\| + \|\mathcal{O}^x\|\, \|O^y\, F - \mathcal{O}^y\, F\|\,.
\end{aligned}
$$

The third line of this inequality shows that we can still get the exponential convergence of two dimensional Sinc approximation, via one dimensional Sinc operators, provided that $F \in \mathbf{X}^2$, i.e., provided that there exist positive constants c_1 and c_2, such that

1. For each fixed $y \in [0,1]$, function $F(\cdot\,, y)$ of one variable belongs to the appropriate space $\mathbf{X}^1$ to enable an error of the form

$$
\mathcal{O}\left(\exp(-c_1\, N_1^{1/2})\right)\,;
$$

and

2. The norm $\|O^y\|$ is bounded,

and dually, provided that

1. For each fixed $x \in [0,1]$, the one dimensional function $F(x\,,\cdot)$ satisfies the appropriate one-dimensional conditions to enable an error of the form

$$\mathcal{O}\left(\exp(-c_2\, N_2^{1/2})\right);$$

   and

2. The norm $\|\mathcal{O}^x\|$ is bounded.

The above space $\mathbf{X}^2$ is thus an appropriate one for Sinc approximation.

VI. (a) DIRICHLET PROBLEMS IN TWO DIMENSIONS.

We discuss here the problem

$$\nabla^2\, u(\bar{r}) = 0\,, \quad \bar{r} \in \mathcal{B}^2,$$

$$u(\bar{r}) = g(\bar{r})\,,\ \bar{r} \in \mathcal{B}^1 = \partial\mathcal{B}^2\,,$$

where the contour $\mathcal{B}^1$ and the two dimensional region $\mathcal{B}^2$ are defined above.

Let us denote the Hilbert transform of F taken over $\mathcal{B}^1$ by

$$(\mathcal{S}F)(\zeta) = \frac{P.V.}{\pi\, i}\int_{\mathcal{B}^1}\frac{F(t)}{t-\zeta}\,dt\,.$$

Let v denote a conjugate harmonic function of the solution u to (3.9), and let $f = u + iv$ denote a function that is analytic in

$$D = \left\{ z = x + i\, y \in \mathbb{C} : (x, y) \in \mathcal{B}^2 \right\}$$

.

Let us set

$$f(z) = \frac{1}{\pi\, i} \int_{\mathcal{B}^1} \frac{\mu(\tau)}{\tau - z}\, d\tau$$

where μ is a real valued function on $\mathcal{B}^1$, which is to be determined.

Upon letting $z \to \zeta \in \mathcal{B}^1$, with ζ not a corner point of $\mathcal{B}^1$, and taking real parts, we get the equation

$$\mu(\zeta) + (K\, \mu)(\zeta) = g(\zeta)\,, \tag{1}$$

with $K\mu = \Re \mathcal{S}\, \mu$.

The integral equation operator K defined by $K\, u = \Re\, \mathcal{S}\, u$ arises for nearly every integral equation that is used for constructing the conformal maps. It has been shown in [Gaier, 1964] that this operator K has a simple eigenvalue 1, for which the corresponding eigenfunction is also 1. Furthermore, the other eigenvalues λ such that the equation $K\, v = \lambda\, v$ has non–trivial solutions v are all less than 1 in absolute value.

Writing $\kappa = (K - 1)/2$ we can rewrite (1) as follows:

$$\mu(\zeta) + (\kappa\, \mu)(\zeta) = g(\zeta)/2\,.$$

Since the norm of κ is less than one in magnitude, the series

$$\sum_{p=0}^{\infty} (-1)^p \, \kappa^p \, g$$

converges to the unique solution of the integral equation. Since $g \in \mathbf{Lip}_\alpha(\mathcal{B}^1)$, we know that (see [Gakhov]) $\mathcal{S}\, g \in \mathbf{Lip}_\alpha(\mathcal{B}^1)$, so that $\kappa\, g \in \mathbf{Lip}_\alpha(\mathcal{B}^1)$. It thus follows, that $\kappa : \mathbf{X}^1 \to \mathbf{X}^1$. The series sum thus converges to a function $\mu \in \mathbf{X^1}$.

It thus follows that the one dimensional methods of §2 can be used to solve two dimensional Dirichlet problems, i.e., we have obtain a solution via simple one dimensional methods, i.e., via separation of variables.

VI. (b) NEUMANN PROBLEMS IN TWO DIMENSIONS.

The problem takes the form

$$\nabla^2 \, v(\bar{r}) = 0 \, , \quad \bar{r} \in \mathcal{B}^2,$$

$$\frac{\partial v}{\partial \mathbf{n}} = \gamma \quad \text{on} \quad \gamma = \mathcal{B}^1 = \partial \mathcal{B}^2 \, ,$$

where $\mathbf{n}$ denotes the unit outward normal at points of smoothness of $\mathcal{B}^1$, Let γ_J denote the restriction of γ to $\mathcal{B}^1_J$ and set $\gamma_J(t) = \gamma_J(\bar{\rho}_J(t))$, where $\bar{\rho}_J$ is defined as above. Then we have $\gamma_J \in \mathbf{M}_{\alpha,d}(\varphi)$. If v denotes the solution to Neumann's problem, and if u denotes the function conjugate to v, we have $u = \mathcal{S}\, v$, and also, $v = \mathcal{S}\, u$, where $\mathcal{S}$ denotes the Hilbert transform defined above Furthermore, the Cauchy–Riemann equations imply that

$$\frac{\partial v}{\partial \mathbf{n}} = -\frac{\partial u}{\partial \mathbf{t}} = -\gamma\,,$$

where $\mathbf{t}$ denotes the unit tangent at points of smoothness of $\mathcal{B}^1$. Given γ, we can thus determine $g(z) = \int_a^z h(t)\,dt$, where the integrations are taken along $\mathcal{B}^1$, and where we can accurately carry out such indefinite integrations via use of Sinc indefinite integration along each segment $\mathcal{B}^1_J$ of $\mathcal{B}^1$. We can thus solve a Dirichlet problem to determine a function ν on $\mathcal{B}^1$, as we did above to determine μ, and having determined ν, we can get $\mu = \mathcal{S}\,\nu$ via the Sinc approximation procedure for approximating the Hilbert transform. We can then approximate v in the interior of $\mathcal{B}^2$ via the above procedure for solving a Dirichlet problem.

VI. (c) Solution of a Poisson Problem on $\mathcal{B}^2$

A particular solution to Poisson's equation

$$\nabla^2\, w(\bar{r}) = -e(\bar{r})\,, \quad \bar{r} \in \mathcal{B}^2\,,$$

is given by

$$w(\bar{r}) = \int_{\mathcal{B}^{(2)}} \mathcal{G}(x-\xi\,,\; y-\eta)\, e(\xi\,,\eta)\, d\xi\, d\eta\,,$$

where $\bar{r} = (x\,,y)$, and where $\mathcal{G}$ denotes the Green's function

$$\mathcal{G}(x,y) = \frac{1}{2\,\pi}\log\left\{\frac{1}{\sqrt{x^2+y^2}}\right\}\,.$$

Theorem: *If* $f \in \mathbf{X}^2(\mathcal{B}^2)$, *then* $w \in \mathbf{X}^2(\mathcal{B}^2)$.

Proof §6.5 of [Stenger, 1993].

Example: Let $\mathcal{B}^2$ denote the region

$$\mathcal{B}^2 = \left\{(x, y) : -1 < x < 1\,, \quad -\sqrt{1-x^2} < y < \sqrt{1-x^2}\right\}\,,$$

i.e., $\mathcal{B}^2$ is the circular region with boundary $B^1 = \cup_{j=1}^2 \Gamma_j\,,$ where Γ_1 is the upper semicircular boundary of radius 1, and Γ_2 is the lower semi-circular boundary of radius 1. We consider the solution to the problem

$$u_{xx} + u_{yy} = -e(x, y) = -4, \quad (x, y) \in \mathcal{B}^2$$

$$u(x, y) = \begin{cases} \;\;1 & \text{if} \quad (x, y) \in \Gamma_1 \\ -1 & \text{if} \quad (x, y) \in \Gamma_2\,. \end{cases}$$

A particular solution to the non-homogeneous problem is

$$U(x, y) = \int\int_{x^2+y^2<1} \mathcal{G}(x-\xi, y-\eta)\, e(\xi, \eta)\, d\xi\, d\eta = (1-x^2-y^2)\,,$$

(See "lap_poi_disc.m") whereas

$$v(x, y) = \frac{2}{\pi}\; \arctan\left(\frac{2\,y}{1-x^2-y^2}\right)\,,$$

solves the homogeneous problem with the boundary conditions. (See "lap_harm_disc.m") .

VII. TIME PROBLEMS

Consider, for example, the integral equation solution

$$U(x, y, t) = \int_0^T \int\int_{\mathcal{B}} \mathcal{G}(x-\xi, y-\eta, t-t')\, f(\xi, \eta, t')\, d\xi\, d\eta\, dt'$$

which satisfies the equation

$$\frac{1}{c^2} U_{tt} - U_{xx} - U_{yy} = f(x,y,t), \quad (x,y,t) \in \mathcal{B} \times (0,T)\,,$$

and with $\mathcal{G} = \mathcal{G}(x,y,t)$ the free–space Green's function which satisfies the equation

$$\frac{1}{c^2} \mathcal{G}_{tt} - \mathcal{G}_{xx} - \mathcal{G}_{yy} = \delta(t)\,\delta(x)\,\delta(y)\,.$$

The above integral expression for $U(x,y,t)$ (even when the right hand side is a nonlinear function that may depend not only on (x,y,t) , but also on U_x , U_y , and U_t) can of course be solved iteratively, by successive approximation, for $U(x,y,t)$ over $\mathcal{B} \times (0,T)$, provided that T is sufficiently small. We can similarly also obtain an approximate solution over this region by doing successive approximation based on Sinc convolution, via use of the "Laplace transform"

$$\begin{aligned} & G(u,v,\tau) \\ & \quad = \int_0^\infty \int_0^\infty \int_0^\infty G(x,y,t) \exp\left(-\frac{x}{u} - \frac{y}{v} - \frac{t}{\tau}\right) dt\, dx\, dy\,, \end{aligned}$$

In "Sinc-Pack", one finds an explicit expression for this "Laplace transform":

$$\begin{aligned} & G(u,v,\tau) \\ & \quad = \left(\frac{1}{c^2\tau^2} - \frac{1}{u^2} - \frac{1}{v^2}\right)^{-1} \left(\frac{1}{4} - H(u,v,\tau) - H(v,u,\tau)\right)\,, \end{aligned}$$

where

$$H(u,v,\tau) = \frac{1}{u}\frac{1}{\sqrt{\frac{1}{c^2\tau^2} - \frac{1}{v^2}}} \,.$$

In the solution procedure, one first determines the spacial accuracy, before determining the time accuracy. This means that we first select the spacial Sinc indefinite integration matrices, which fixes the eigenvalues (that occupy the positions (u,v) in the above expression for $G(u,v,\tau)$) and eigenvectors of these matrices. The corresponding eigenvalues of Sinc time indefinite integration matrices for integration over $(0,T)$ are just T times the corresponding eigenvalues for integration over $(0,1)$, while the eigenvectors of these matrices are the same as those for integration over $(0,1)$. It thus follows, upon replacing τ by $T\,\tau$ in the above expression for $G(u,v,\tau)$, then keeping u , v , and τ fixed, that $G(u,v,T\,\tau) \to 0$ as $T \to 0$. Gersgorin's theorems (see *Gersgorin and His Circles*, by R.S. Varga) may thus be used to prove that we have a contraction for all T sufficiently small, i.e., that the procedure converges, even for nonlinear wave problems.

Similar statements can be made for the case of wave problems over $\mathcal{B} \times (0,T)$, with $\mathcal{B} \subseteq \mathbb{R}^3$, and for heat problems over $\mathcal{B} \times (0,T)$.