Experimental approximations with shifts of $\exp(az)/z$

"Complex sinc function"

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Haar spaces

Definition

A finite dimensional subspace H of $X := \mathrm{C}(D)$ with dimension n is called a *Haar space*, if its non trivial members possess at most $n-1$ zeros in D.

Example

Let

$$f := 1/t \text{ for all } t \in \mathbb{C}\backslash\{0\}$$

and let D be any compact set in $\mathbb{C}$ and $S := \complement D$ with respect to $\mathbb{C}$, where $\complement$ stands for the complement. Let $s_j \in S$, $j \in \mathbb{N}$ any sequence of pairwise distinct points. Then for all n, the spaces

$$H_n := \langle h_1, h_2, \ldots, h_n \rangle, \quad h_j(t) := f(t - s_j), j \in \mathbb{N}$$

define Haar spaces with $H_n \subset H_{n+1}$. This is easy to see: A typical element of H_n is of the form

Example (contd.)

$$\eta_n := \sum_{j=1}^{n} a_j h_j, \quad a_j \in \mathbb{C}.$$

This expression can be given the form

$$\eta_n(t) = \sum_{j=1}^{n} a_j h_j(t) =$$

$$\frac{1}{\prod_{j=1}^{n}(t - s_j)} \sum_{j=1}^{n} a_j \prod_{k \neq j} (t - s_k) =: \frac{p(t)}{q(t)},$$

Example (contd.)

where $p \in \Pi_{n-1}, q \in \Pi_n$. Since q has no zeros in D, η_n is well defined and η_n has at most $n-1$ zeros in D, since p has at most $n-1$ zeros in $\mathbb{C}$, provided p is non-trivial.

In this example

$$f(t) := 1/t$$

universally generates Haar spaces in the real and in the complex case.

Haar spaces

Example

Let $f(t) := \exp(-t^2)$, $t \in \mathbb{K}$. We consider spaces spanned by $h_j(t) := f(t - s_j)$ with arbitrary but pairwise distinct shifts $s_j \in \mathbb{K}$, $j \in \mathbb{N}$. We have $h_j(t) = \exp(-s_j^2)\exp(-t^2)\exp(2s_j t)$. Thus, for all $n \in \mathbb{N}$ we have

$$H_n := \langle h_1, h_2, \ldots, h_n \rangle =$$

$$\exp(-t^2)\langle \exp(2s_1 t), \exp(2s_2 t), \ldots, \exp(2s_n t)\rangle.$$

Since e^z has no zeros for all $z \in \mathbb{C}$, the problem is reduced to the investigation of the space

$$\tilde{H}_n := \langle \exp(s_1 t), \exp(s_2 t), \ldots, \exp(s_n t)\rangle.$$

If $\mathbb{K} = \mathbb{R}$ the space $\tilde{H}_n$ belongs in the catalogue of well known

Example (contd.)

Haar spaces, Karlin & Studden, [1966, Example 1, p. 9-10]. If, however, $\mathbb{K} = \mathbb{C}$ the spaces $\tilde{H}_n$ are no longer Haar spaces for $n \geq 2$.

In this example

$$f(t) := \exp(-t^2)$$

does not universally generate Haar spaces in the complex case.

Universal Haar space generators

We will denote by $\mathbb{D}$ the open, unit disk in $\mathbb{C}$ and correspondingly $D := \overline{\mathbb{D}}$ will denote the closed, unit disk.

Definition

Let $n \in \mathbb{N}$ be a fixed natural number. A function G defined on $\mathbb{C}\setminus\{0\}$ with values in $\mathbb{C}$ will be called an *n-dimensional Haar space generator*, if for each set of n pairwise distinct points $s_1, s_2, \ldots, s_n \in \mathbb{C}\setminus\overline{\mathbb{D}}$, the functions h_j defined by $h_j(z) := G(z - s_j)$, $j = 1, 2, \ldots, n$ for $z \in \overline{\mathbb{D}}$ span an n-dimensional Haar space.

Example

Let $G(z) := z^{m-1}$ with $m \geq 1$ fixed. Then G is an m-dimensional Haar space generator but not an $(m+1)$-dimensional Haar space generator.

Universal Haar space generators

Definition

A function G defined on $\mathbb{C}\backslash\{0\}$ with values in $\mathbb{C}$ will be called a *universal Haar space generator*, if for each $n \in \mathbb{N}$, it is an n-dimensional Haar space generator. A universal Haar space generator will be abbreviated by *UHG*.

Theorem (Hauptsatz)

Let G be analytic on $\mathbb{C}\backslash\{0\}$. Then G is a UHG if and only if G is of the form

$$G(z) := \frac{\mathrm{e}^{az+b}}{z}, \quad a \in \mathbb{C}, b \in \mathbb{C}. \tag{1}$$

The proof consists of requiring that G is a one dimensional Haar space generator, a one and a two dimensional Haar space generator etc.

Universal Haar space generators

Lemma

Let G be analytic in $\mathbb{C}\backslash\{0\}$ and let G generate Haar spaces of dimension one. Then, G does not vanish on $\mathbb{C}\backslash\{0\}$.

This next result is obtained by assuming that G generates one and two dimensional Haar spaces.

Lemma

Let G be analytic in $\mathbb{C}\backslash\{0\}$ and let G generate Haar saces of dimension one and of dimension two. Then, G has the form

$$G(z) := z^m \mathrm{e}^{\phi(z)}, \quad m \in \mathbb{Z}, \tag{2}$$

where ϕ is an entire function.

Universal Haar space generators

By Theorem Hauptsatz we also can identify many non-Haar spaces. One set of examples is produced by

$$g(z) := \frac{1}{z^m}, \; m \geq 2$$

and the corresponding spaces generated by shifts. For $m = 2$ this was shown by RAHMAN & RUSCHEWEYH [1983].
The main Theorem (Hauptsatz) is also valid for (non empty) compact sets $D \subset \mathbb{C}$.

The theoretical part is joint work with the late Walter Hengartner, *1936 (St. Gallen) – †2003 (Québec City):

- Constr. Approx. **22** (2005), 113–132,
- Comput. Methods Funct. Theory **3** (2003), 151–164,
- Approximation Theory 10, Abstract and Classical Analysis, Vanderbilt University Press, Nashville, 2002, 223–238 (eds.: C. K. Chui, L. L. Schumaker, J. Stöckler).

Examples

Let us treat the interpolation problem in the space

$$V_n := \langle v_1, v_2, \ldots, v_n \rangle, \text{ where } v_j(z) := \exp(a(z - s_j)) \frac{1}{z - s_j}$$

with given constant $a \in \mathbb{C}$ and given shifts $s_j \in \mathbb{C}$, $j = 1, 2, \ldots, n$. That means, we want to find the unique element $v \in V_n$ which satisfies (f_j given)

$$v(z_j) = f_j,\ j = 1, 2, \ldots, n. \tag{3}$$

A typical element $v \in V_n$ has the form

$$v(z) := \frac{\exp(az)}{q(z)} p(z), \text{ where } q(z) := \prod_{j=1}^{n} (z - s_j),\ p \in \Pi_{n-1}. \tag{4}$$

Thus, we need to determine p of (4). Because of (3), this is equivalent to

Examples

$$p(z_j) = f_j q(z_j) \exp(-az_j) =: \eta_j,\ j = 1, 2, \ldots, n. \qquad (5)$$

Now, this is a simple polynomial interpolation problem

$$p(z_j) = \eta_j,\ j = 1, 2, \ldots, n.$$

Having found $p \in \mathbf{\Pi}_{n-1}$, the wanted solution is v according to (4), namely

$$v := \frac{\exp(az)}{q(z)} p(z), \quad q(z) := \prod_{j=1}^{n} (z - s_j).$$

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Examples

We choose z_j equidistantly on the unit circle,

$$z_j := \exp(\frac{2\pi \mathbf{i} j}{n}),\, j = 0, 1, \ldots, n-1 \text{ and } f_j := f(z_j),$$

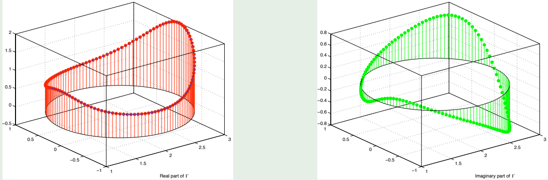
where

$$f := \frac{1}{\Gamma}$$

is the inverse of the complex gamma function. It is an entire function, thus, analytic in all of $\mathbb{C}$. The next step is to choose the constant a, the number n, and the shifts $s_1, s_2, \ldots, s_n$.

We first show some pictures of Γ and $1/\Gamma$ whose values on the unit circle determine the functions completely.

Example

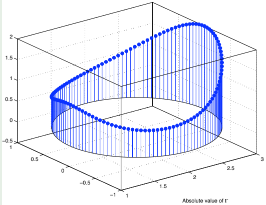


Figure

Real (left) and imaginary part of Γ function

Examples

Example (contd.)

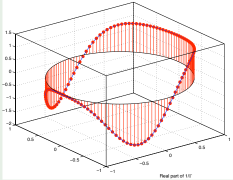


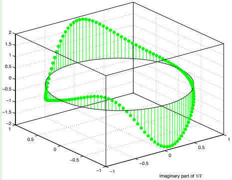
Figure

Absolute value of Γ function

Examples

Example

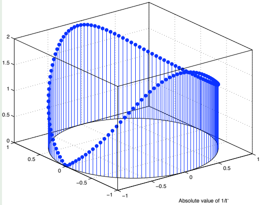




Figure

Real (left) and imaginary part of $1/\Gamma$ function

Example (contd.)



Figure

Absolute value of $1/\Gamma$ function

Example

We selected the following four shifts

$$s_1 := -3.67, s_2 := -3.57, s_3 := 3.57, s_4 := 3.67$$

and chose

$$a := -1.009.$$

The result can be viewed in the next Figure, left side. Interestingly, the maximal error $\max_{|z|=1} |(f - v)(z)|$ depends sharply on the constant a, to be viewed in next Figure, right side.

In comparison, the corresponding interpolating polynomial of degree three has a much larger error.

Examples # 1

Example (contd.)

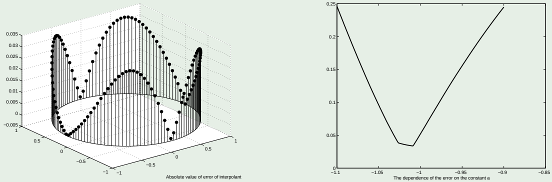


Figure (Frank Stenger crown 1)

Absolute error $|f - v|$ of interpolant v (left) and dependence of the maximal absolute error on the constant a occurring in (4) (right)

Examples # 2

Example

In the next example we selected ten shifts

$$\pm 3.3, \pm 4.3, \pm 5.3, \pm 6.3, \pm 7.3,$$

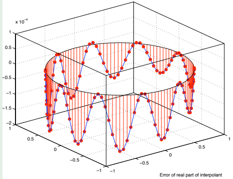
and

$$a := -0.5.$$

The results are shown in the next figures.

Examples # 2

Example (contd.)



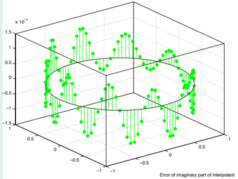
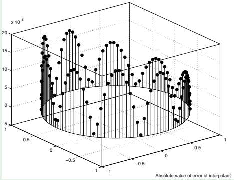


Figure (Frank Stenger crowns 2 and 3)

Real (left) and imaginary part of $1/\Gamma - v$ of interpolant v

Examples # 2

Example (contd.)



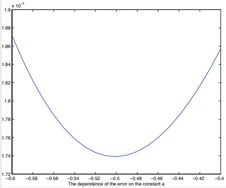


Figure (Frank Stenger crown 4)

Absolute error $|1/\Gamma - v|$ of interpolant v (left) and dependence of the maximal absolute error on the constant a occurring in (4) (right)

Summary

1. Interpolation: very simple,
2. Approximation: not too difficult,
3. Shift selection: no results, so far just experimental,
4. Parameter selection: no results, just experimental.
5. Thank You!

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