Recent Developments in Variable Transformations for Numerical Integration

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Consider the problem of evaluating finite-range integrals of the form

$$I[f] = \int_0^1 f(x)\,dx,$$

with $f \in C^\infty(0,1)$ but not necessarily continuous or differentiable at $x = 0$ and $x = 1$. $f(x)$ may even behave singularly at the endpoints, with different types of singularities. One simple and effective way of computing $I[f]$ is by first transforming it with a suitable variable transformation and next applying the trapezoidal rule to the resulting transformed integral. Thus, if we make the substitution $x = \psi(t)$, where $\psi(t)$ is an increasing differentiable function on $[0,1]$, such that $\psi(0) = 0$ and $\psi(1) = 1$, then the transformed integral is

$$I[\widehat{f}\,] = \int_0^1 \widehat{f}(t)\,dt; \quad \widehat{f}(t) = f(\psi(t))\psi'(t).$$

If $\psi(t)$ is chosen such that $\psi^{(i)}(0) = \psi^{(i)}(1) = 0$, $i = 1, 2, \dots, p$, for some sufficiently large p, then $Q_n[\widehat{f}\,]$, the n-panel trapezoidal rule approximation to $I[\widehat{f}\,]$, approximates $I[\widehat{f}\,]$ with surprisingly high accuracy even for moderate n.

Variable transformations in numerical integration have been of considerable interest lately. In the context of one-dimensional integration, they are used as a means to improve the performance of the trapezoidal rule. Recently, they have also been used to improve the performance of the Gauss–Legendre quadrature. In the context of multi-dimensional integration, they are used to "periodize" the integrand in all variables so as to improve the accuracy of lattice rules. (Lattice rules are extensions of the trapezoidal rule to many dimensions.) They have also been used for computing integrals on smooth surfaces in $\mathbb{R}^3$.

In this lecture, we give an overviw of the subject. We discuss the various transformations, old and new. We present some recent developments that enable one to obtain "optimal" numerical results from some transformations that have power-like behavior at the endpoints.