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INTERVAL ANALYSIS I

by

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INTRODUCTION

Digital computations by computers consist of finite sequences of psuedo-arithmetic operations. On the other hand, the exact numerical solution of a mathematical problem, if computable at all, often requires an infinite sequence of exact arithmetic operations.

The study of approximation by digital computations is the underlying motivation for the present study. A digital computation and the analysis of its error as an approximation are usually carried out separately. However, in the present study an interval arithmetic is devised which forms a basis for a concomitant analysis of error in a digital computation. In this system, computations are performed with intervals and intervals are so produced to contain, by construction, the exact numerical solutions sought. Hence an approximation and its possible error will be obtained at the same time, choosing say the midpoint of an interval as the approximation.

This report is the first part of our study, in which we first examine some properties of exact or ideal interval arithmetic. After a preliminary discussion of the space of intervals (§ 1) we study addition, multiplication, subtraction and division of intervals (§§ 2-5). Then we construct arithmetic functions as compositions of these elementary operations (§ 6). As one may expect, arithmetic functions play exactly the same role in interval analysis as rational functions in real analysis, so that there is a relation between arithmetic functions and rational functions (§ 7).

In the present report we apply interval analysis to the study of the following problems. Let $f(x)$ be a continuous real-valued function defined on an interval $[a,b]$. (1) What is the image interval $f([a,b])$? (2) What is the definite integral $\int_a^b f(x)\,dx$? When $f(x)$ is a rational function, the approximations are given in theorems 1 and 2 (§§ 8,9). In general, if $f(x)$ is an arbitrary continuous function, we may still have approximation theorems 3 and 4 (§ 10), although they are not as precise as the first two theorems.

In forthcoming reports we shall apply interval analysis to differential equations and report on results of machine computations using a digital version of interval arithmetic modified to enable the computations to be carried out with pseudo-arithmetic operations. (See also, LMSD-48421, "Automatic Error Analysis in Digital Computation," by R. E. Moore.)

1. PRELIMINARIES

Throughout the whole study R denotes the real line. Whenever a and b are real numbers with $a \leqslant b$, $[a,b]$ denotes the subset of R consisting of all the real numbers x with $a \leqslant x \leqslant b$. In symbols,

$$[a,b] = \{x \in R \mid a \leqslant x \leqslant b\} .$$

Let $\mathcal{I}$ be the set of all such sets $[a,b]$.

That means,

$$\mathcal{I} = \{[a,b] \mid a,b \in R \text{ and } a \leqslant b\} .$$

Then we have natural functions

$$p : R \longrightarrow \mathscr{I} ,$$
$$\alpha : \mathscr{I} \longrightarrow R ,$$
$$\beta : \mathscr{I} \longrightarrow R ,$$
$$\gamma : \mathscr{I} \longrightarrow R ,$$
$$\sigma : \mathscr{I} \longrightarrow R ,$$

defined by

$$p(x) = [x,x] ,$$
$$\alpha([a,b]) = a ,$$
$$\beta([a,b]) = b ,$$
$$\gamma([a,b]) = \max \{|a| , |b|\} ,$$
$$\sigma([a,b]) = b - a$$

respectively, where $x \epsilon R$ and $[a,b] \epsilon \mathscr{I}$.

Whenever $x,x' \epsilon R$, we let

$$\rho(x,x') = |x - x'| .$$

Then ρ is a metric on R , that means, ρ has the following properties.

(1) Whenever $x,x' \epsilon R$, $\rho(x,x') = 0$ if and only if $x = x'$.

(2) Whenever $x,x' \epsilon R$, $\rho(x,x') = \rho(x',x)$.

(3) Whenever $x,x',x'' \epsilon R$, $\rho(x,x') + \rho(x',x'') \geqslant \rho(x,x'')$.

Whenever $A, A' \in \mathcal{A}$ we let

$$P(A,A') = \max \left\{ \rho\Big(\alpha(A) \ , \ \alpha(A')\Big) \ , \ \rho\Big(\beta(A) \ , \ \beta(A')\Big) \right\} .$$

Then P has the same properties as those ρ has on R so that P is a metric on $\mathcal{A}$.

As direct consequences of the definitions of ρ , P , α , β , γ , σ we have

(1-1) The function $p : R \rightarrow \mathcal{A}$ is isometric; that means, for any $x, x' \in R$,

$$P\Big(p(x) \ , \ p(x')\Big) = \rho(x,x') .$$

Hence p maps R homeomorphically onto $p(R)$.

(1-2) Whenever $A, A' \in \mathcal{A}$,

$$\rho\Big(\alpha(A) \ , \ \alpha(A')\Big) \leq P(A,A') .$$

Hence the function $\alpha : \mathcal{A} \longrightarrow R$ is uniformly continuous, that means, for any $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $A, A' \in \mathcal{A}$ with $P(A,A') < \delta$, we have $\rho\Big(\alpha(A) \ , \ \alpha(A')\Big) < \varepsilon$.

Since uniformly continuous functions are continuous, it follows that:

(1-3) The function $\alpha : \mathcal{A} \rightarrow R$ is continuous, that means, for any $A \in \mathcal{A}$ and any $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $A' \in \mathcal{A}$ with $P(A,A') < \delta$ we have $\rho\Big(\alpha(A) \ , \ \alpha(A')\Big) < \varepsilon$.

Just as (1-2) and (1-3), we have

(1-4) Whenever $A, A' \in \mathcal{A}$,

$$\rho\Big(\beta(A) \ , \ \beta(A')\Big) \leqslant P(A,A') .$$

Hence the function $\beta:\mathcal{S} \longrightarrow R$ is uniformly continuous and consequently it is continuous.

Using (1-2), (1-4) and well-known properties of real numbers, we have

(1-5) The functions $\gamma : \mathcal{S} \longrightarrow R$ and $\sigma : \mathcal{S} \longrightarrow R$ are uniformly continuous and hence they are continuous.

The following will be needed later.

(1-6) Let $A, A' \epsilon \mathcal{S}$ and let $\alpha > 0$. Then $P(A,A') < \alpha$ if and only if the following two conditions hold.

(i) For every $x \epsilon A$ there is some $x' \epsilon A'$ with $\rho(x,x') < \alpha$.

(ii) For every $y' \epsilon A'$ there is some $y \epsilon A$ with $\rho(y',y) < \alpha$.

Whenever $I \epsilon \mathcal{S}$ we let

$$\mathcal{S}_I = \{A \epsilon \mathcal{S} \mid A \subset I\} \quad .$$

(1-7) Whenever $I \epsilon \mathcal{S}$, $\mathcal{S}_I$ is compact; that means, every sequence in $\mathcal{S}_I$ contains a convergent subsequence.

Let $\mathcal{J}$ be the subset of $\mathcal{S}$ consisting of all the elements of $\mathcal{S}$ <u>not</u> containing 0 .

(1-8) The set $\mathcal{J}$ is open in $\mathcal{S}$; that means, for every $A \epsilon \mathcal{J}$ there is a positive number γ_A such that every $A' \epsilon \mathcal{S}$ with $P(A,A') < \gamma_A$ belongs to $\mathcal{J}$. In fact, we may choose $\gamma_A = \min \{|\alpha(A)|, |\beta(A)|\}$.

2. ADDITION

There is a function

$$\oplus : \mathscr{I} \times \mathscr{I} \longrightarrow \mathscr{I}$$

defined by

$$\oplus\,(A,B) = \{x + y \mid x \in A \text{ and } y \in B\}$$

$$= [\alpha(A) + \alpha(B)\ ,\ \beta(A) + \beta(B)]\ ,\ A,B \in \mathscr{I}\ .$$

The set $\oplus(A,B)$ is also written $A \oplus B$. The function $\oplus$ is called the addition on $\mathscr{I}$.

(2-1) The function $p : R \rightarrow \mathscr{I}$ preserves the addition; that means, for any $x,y \in R$,

$$p\,(x) \oplus p\,(y) = p\,(x + y)\ .$$

Because of (2-1), the addition $\oplus$ on $\mathscr{I}$ may be regarded as an extension of the addition $+$ on R . This is the reason for calling $\oplus$ the "addition" on $\mathscr{I}$.

(2-2) The addition $\oplus$ is commutative; that means, for any $A,B \in \mathscr{I}$,

$$A \oplus B = B \oplus A\ .$$

(2-3) The addition $\oplus$ is associative; that means, for any $A,B,C \in \mathscr{I}$,

$$A \oplus (B \oplus C) = (A \oplus B) \oplus C\ .$$

(2-4) Whenever $(A,B)\ ,\ (A',B') \in \mathscr{I} \times \mathscr{I}$,

$$P(A \oplus B\ ,\ A' \oplus B') \leqslant P(A,A') + P(B,B')\ .$$

Hence the addition $\oplus$ is uniformly continuous; that means, for any $\mathcal{E} > 0$ there is a $\delta > 0$ such that whenever $(\mathbf{A},B)$, $(\mathbf{A}',B') \in \mathcal{J} x \mathcal{J}$ with $P(\mathbf{A},\mathbf{A}') < \delta$ and $P(B,B') < \delta$, we have $P(\mathbf{A} \oplus B, \mathbf{A}' \oplus B') < \mathcal{E}$.

Proof. Let

$$A = [a,b] , \mathbf{A}' = [a',b'] , B = [c,d] , B' = [c',d'] .$$

Then

$$P(\mathbf{A} \oplus B , \mathbf{A}' \oplus B') = P\Big([a + c , b + d] , [a' + c' , b' + d']\Big)$$

$$= \max \{ \rho(a + c , a' + c') , \rho(b + d , b' + d')\}$$

$$\leqslant \max \{ \rho(a,a') + \rho(c,c') , \rho(b,b') + \rho(d,d')\}$$

$$\leqslant \max \{ \rho(a,a') , \rho(b,b')\} + \max \{ \rho(c,c') , \rho(d,d')\}$$

$$= P(\mathbf{A},\mathbf{A}') + P(B,B') .$$

To prove the uniform continuity we have only to pick $\delta = \mathcal{E}/2$. q.e.d.

Since uniformly continuous functions are continuous, it follows that

(2-5) The addition $\oplus$ is continuous; that means, for any $(\mathbf{A},B) \in \mathcal{J} x \mathcal{J}$ and any $\mathcal{E} > 0$ there is a $\delta > 0$ such that whenever $(\mathbf{A}',B') \in \mathcal{J} x \mathcal{J}$ with $P(\mathbf{A},\mathbf{A}') < \delta$ and $P(B,B') < \delta$, we have $P(\mathbf{A} \oplus B, \mathbf{A}' \oplus B') < \mathcal{E}$.

3. MULTIPLICATION

Whenever $\mathbf{A},B \in \mathcal{J}$ we let

$$\otimes(\mathbf{A},B) = \{xy \mid x \in \mathbf{A} \text{ and } y \in B\} .$$

We claim that $\otimes(\mathbf{A},B) \in \mathcal{J}$.

Let $A = [a,b]$ and $B = [c,d]$. Then we have the following cases

(1) If $a \geqslant 0$ and $c \geqslant 0$, then

$$\otimes (A,B) = [ac,bd] .$$

(2) If $a \geqslant 0$ and $c < 0 < d$, then

$$\otimes (A,B) = [bc,bd] .$$

(3) If $a \geqslant 0$ and $d \leqslant 0$, then

$$\otimes (A,B) = [bc,ad] .$$

(4) If $a < 0 < b$ and $c \geqslant 0$, then

$$\otimes (A,B) = [ad,bd] .$$

(5) If $a < 0 < b$ and $c < 0 < d$, then

$$\otimes (A,B) = [\min \{bc,ad\} , \max \{ac,bd\}] .$$

(6) If $a < 0 < b$ and $d \leqslant 0$, then

$$\otimes (A,B) = [bc,ac] .$$

(7) If $b \leqslant 0$ and $c \geqslant 0$, then

$$\otimes (A,B) = [ad,bc] .$$

(8) If $b \leqslant 0$ and $c < 0 < d$, then

$$\otimes (A,B) = [ad,ac] .$$

(9) If $b \leqslant 0$ and $d \leqslant 0$, then

$$\otimes (A,B) = [bd,ac] .$$

Because of this result, we have a function

$$\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$$

called the <u>multiplication</u> on $\mathcal{A}$. Whenever $A,B \in \mathcal{A}$, the set $\otimes(A,B)$ is also written $A \otimes B$.

(3-1) The function $p : R \longrightarrow \mathcal{A}$ preserves the multiplication.

Because of (3-1) the multiplication $\otimes$ on $\mathcal{A}$ may be regarded as an extension of the multiplication on R.

(3-2) The multiplication $\otimes$ is commutative; that means, for any $A,B \in \mathcal{A}$,

$$A \otimes B = B \otimes A .$$

(3-3) The multiplication $\otimes$ is associative; that means, for any $A,B,C \in \mathcal{A}$,

$$A \otimes (B \otimes C) = (A \otimes B) \otimes C$$

(3-4) For any $A,B,C \in \mathcal{A}$,

$$A \otimes (B \oplus C) \subset (A \otimes B) \oplus (A \otimes C)$$

but both sides may not be equal. Hence the distributive law does not hold.

(3-5) Whenever (A,B), $(A',B') \in \mathcal{A} \times \mathcal{A}$

$$P(A \otimes B , A' \otimes B') \leqslant \gamma(B) \; P(A,A') + \gamma(A') \; P(B,B') .$$

Hence the multiplication is continuous.

Proof. Let $xy \in A \otimes B$, where $x \in A$ and $y \in B$. By (1-6), there is some $x' \in A'$ and some $y' \in B'$ such that $\rho(x,x') \leqslant P(A,A')$ and $\rho(y,y') \leqslant P(B,B')$.

Therefore $x'y' \epsilon A' \otimes B'$ and

$$\rho(xy, x'y') = |xy - x'y'|$$
$$= |y(x - x') + x'(y - y')|$$
$$\leqslant |y| \cdot |x - x'| + |x'| \cdot |y - y'|$$
$$\leqslant \gamma(B) P(A,A') + \gamma(A') P(B,B') .$$

Let $x'y' \epsilon A' \otimes B'$, where $x' \epsilon A'$ and $y' \epsilon B'$. Similarly there is some $xy \epsilon A \otimes B$ such that $x \epsilon A$, $y \epsilon B$ and

$$\rho(x'y', xy) < \gamma(B) P(A,A) + \gamma(A') P(B,B') .$$

Making use of (1-6) again, these results imply the first part of (3-5).

To prove the continuity of $\otimes$, we let $(A,B) \in \mathscr{I} \times \mathscr{I}$ and let $\mathcal{E} > 0$. Take

$$\delta = \min\left\{ \mathcal{E} / \left(\gamma(A) + \gamma(B) + 1 \right), 1 \right\} .$$

Then for any $(A',B') \in \mathscr{I} \times \mathscr{I}$ with $P(A,A') < \delta$ and $P(B,B') < \delta$ we have

$$P(A \otimes B, A' \otimes B') < \gamma(B)\delta + \gamma(A')\delta \leqslant \gamma(B)\delta + \left(\gamma(A) + \delta\right)\delta$$
$$\leqslant \left(\gamma(A) + \gamma(B) + 1\right)\delta < \mathcal{E} . \qquad \text{q.e.d.}$$

It is not hard to see that the multiplication is <u>not</u> uniformly continuous. However, since the restriction of a continuous function on a compact set is uniformly continuous, it follows from (1-7) and (3-5) that

(3-6) Let I and J be fixed elements of $\mathscr{I}$. Then the multiplication

$$\otimes : \mathscr{I}_I \times \mathscr{I}_J \longrightarrow \mathscr{I}$$

is uniformly continuous. In fact, for any $A, A' \in I$ and $B, B' \in J$ we have

$$P(A \otimes B \,,\, A' \otimes B') < \gamma(J)\, P(A,A') + \gamma(I)\, P(B,B') \,.$$

4. SUBTRACTION

By (3-5) we have

(4-1) Let E be a fixed element of $\mathcal{J}$. Then the function $\otimes_E$ of $\mathcal{J}$ into $\mathcal{J}$, defined by

$$\otimes_E(A) = E \otimes A \,, \qquad A \in \mathcal{J} \,,$$

is uniformly continuous.

In particular, if $E = [-1, -1]$, we have

(4-2) The function $\otimes_{[-1,\,-1]} : \mathcal{J} \longrightarrow \mathcal{J}$, defined by

$$\otimes_{[-1,\,-1]}(A) = [-1, -1] \otimes A \,, \qquad A \in \mathcal{J} \,,$$

is uniformly continuous.

Whenever $A \in \mathcal{J}$, we shall abbreviate $\otimes_{[-1,\,-1]}(A)$ by $-A$. Since $-(-A) = A$, $\otimes_{[-1,\,-1]}$ is a homeomorphism.

Combining the addition $\oplus$ and the function $\otimes_{[-1,\,-1]}$ we define the <u>subtraction</u>

$$\ominus : \mathcal{J} \times \mathcal{J} \longrightarrow \mathcal{J}$$

by

$$\ominus(A,B) = A \oplus (-B) = \{x - y \mid x \in A \text{ and } y \in B\} \,, \quad A, B \in \mathcal{J} \,.$$

The set $\ominus(\mathbf{A},B)$ is also written $\mathbf{A} \ominus B$.

From (2-1) and (3-1), it follows

(4-3) The function $p : R \to \mathcal{A}$ preserves the subtraction.

(4-4) Whenever $\mathbf{A},B \in \mathcal{A}$,

$$(-\mathbf{A}) \otimes B = \mathbf{A} \otimes (-B) = -(A \otimes B) ,$$

$$(-\mathbf{A}) \otimes (-B) = \mathbf{A} \otimes B \quad .$$

From (2-4) and (4-2) it follows

(4-5) The subtraction $\ominus$ is uniformly continuous and hence is continuous. In fact, for any $(\mathbf{A},B)$, $(\mathbf{A}',B') \in \mathcal{A} \times \mathcal{A}$,

$$P(\mathbf{A} \ominus B , \mathbf{A}' \ominus B') \leqslant P(\mathbf{A},\mathbf{A}') + P(B,B') .$$

5. DIVISION

For every $\mathbf{A} = [a,b] \in \mathcal{J}$ (see § 1),

$$\mathbf{A}^{-1} = \left\{ x^{-1} \mid x \in A \right\} = \left[b^{-1}, a^{-1} \right]$$

is in $\mathcal{J}$. Hence we have a function $\tau : \mathcal{J} \to \mathcal{J}$ defined by

$$\tau(\mathbf{A}) = \mathbf{A}^{-1} .$$

(5-1) Whenever $J \in \mathcal{J}$, $\mathcal{A}_J \subset \mathcal{J}$ and for any $\mathbf{A},\mathbf{A}' \in \mathcal{A}_J$,

$$P\left(\mathbf{A}^{-1} , \mathbf{A}'^{-1}\right) \leqslant \gamma\left(J^{-1}\right)^2 P(\mathbf{A},\mathbf{A}') .$$

Hence τ is uniformly continuous on $\mathcal{A}_J$, $J \in \mathcal{J}$, and consequently it is continuous on $\mathcal{J}$.

Proof. Let $\mathbf{A},\mathbf{A}'\in\mathcal{A}_J$. For any $x\in A$ there is, by (1-6), some $x'\in A'$ such that $\rho(x,x') \leqslant P(\mathbf{A},\mathbf{A}')$ so that

$$\rho\left(x^{-1},x'^{-1}\right) = | x^{-1} - x'^{-1} | = | x^{-1} | \, | x'^{-1} | \, x - x' |$$

$$\leqslant \gamma(J)^2 P(\mathbf{A},\mathbf{A}') .$$

Similarly, for any $y'\in\mathbf{A}'$ there is some $y\in A$ such that $\rho\left(y'^{-1},y^{-1}\right) \leqslant \gamma(J)^2 P(\mathbf{A},\mathbf{A}')$. Hence the first part is proved.

To prove the uniform continuity of τ on $\mathcal{A}_J$, $J\in\mathcal{J}$, we let $\mathcal{E} > 0$ and take

$$\delta = \mathcal{E}/\gamma\left(J^{-1}\right)^2 .$$

It is clear that for any $\mathbf{A},\mathbf{A}'\in\mathcal{A}_J$ with $P(\mathbf{A},\mathbf{A}') \leqslant \delta$, we have $P\left(\mathbf{A}^{-1},\mathbf{A}'^{-1}\right) < \gamma\left(J^{-1}\right)^2 \delta = \mathcal{E}$.

To prove the continuity of τ we let $\mathbf{A}\in\mathcal{J}$ and let $\mathcal{E} > 0$. Take a $J\in\mathcal{J}$ such that

$$\alpha(J) < \alpha(\mathbf{A}) \leqslant \beta(\mathbf{A}) < \beta(J) .$$

Then for any $\mathbf{A}'\in\mathcal{A}$ with $P(\mathbf{A},\mathbf{A}') < \min\{\alpha(\mathbf{A})-\alpha(J),\ \beta(J)-\beta(\mathbf{A})\}$, $\mathbf{A}'\in\mathcal{J}$. Hence the continuity of τ on $\mathcal{A}_J$ implies the continuity of τ at $\mathbf{A}$. q.e.d.

Since $\left(A^{-1}\right)^{-1} = A, A\in\mathcal{J}$, τ is a homeomorphism.

Combining the multiplication $\otimes$ and the function τ we define the <u>division</u>

$$\oplus : \mathcal{A} \times \mathcal{J} \rightarrow \mathcal{A}$$

by

$$\oplus\ (A,B) = A \otimes B^{-1} = \left\{ xy^{-1} \mid x\epsilon A \text{ and } y\epsilon B \right\},\ A\epsilon\mathcal{J} \text{ and } B\epsilon\mathcal{J} .$$

The set $\oplus\,(A,B)$ is also written A/B .

Since for any real number $x \neq 0$,

$$p\,(x)^{-1} = p\left(x^{-1}\right) ,$$

it follows from (3-1) that

(5-2) The function p preserves the division $\oplus$.

From (3-6) and (5-1) it follows

(5-3) Whenever $I\epsilon\mathcal{J}$ and $J\epsilon\mathcal{J}$,

$$\oplus\ :\ \mathcal{J}_I \times \mathcal{J}_J \longrightarrow \mathcal{J}$$

is uniformly continuous. In fact, for any (A,B) , $(A',B') \epsilon \mathcal{J}_I \times \mathcal{J}_J$ we have

$$P\,(A/B,A'/B') \leqslant \gamma\left(J^{-1}\right) P\,(A,A') + \gamma\left(J^{-1}\right)^2 \gamma(I)\, P\,(B,B') \quad .$$

Hence $\oplus\ :\ \mathcal{J} \times \mathcal{J} \longrightarrow \mathcal{J}$ is continuous.

6. ARITHMETIC FUNCTIONS

Before giving the definition of an arithmetic function we remark that every arithmetic function has a domain contained in $\mathcal{J}$, an order which is a non-negative integer, and a finite number of parameters which are elements of $\mathcal{J}$. An arithmetic function of order n with parameters $A_1, A_2, \ldots A_m$ is written $F^{(n)}_{A_1 A_2 \ldots A_m}$ or simply F . It is a function from its domain $\mathcal{D}(F)$ to $\mathcal{J}$.

An arithmetic function of order n is defined by induction on n. When $n = 0$, the number m of parameters is either 0 or 1. In the former case it is the identity function $F^{(0)} : \mathcal{I} \rightarrow \mathcal{I}$ given by

$$F^{(0)}(X) = X \quad , \qquad X \in \mathcal{I} \; .$$

In the latter case it is the constant function $F_A^{(0)} : \mathcal{I} \rightarrow \mathcal{I}$ given by

$$F_A^{(0)}(X) = A \quad , \qquad X \in \mathcal{I} \; ,$$

where $\mathbf{A}$ is an arbitrary element of $\mathcal{I}$. Notice that every arithmetic function of order 0 has $\mathcal{I}$ as its domain.

Let n be a positive integer and suppose that arithmetic functions of order $< n$ have been defined. Then every arithmetic function $F^{(n)}_{A_1A_2 \ldots A_m}$ of order n is defined as follows. There is an arithmetic function $F^{(s)}_{A_1A_2 \ldots A_\ell}$ of order s, $0 \leqslant s \leqslant n-1$, with parameters $A_1, A_2, \ldots, A_\ell$, $0 \leqslant \ell \leqslant m$, and an arithmetic function $F^{(n-1-s)}_{A_{\ell+1}A_{\ell+2} \ldots A_m}$ of order $n-1-s$ with parameters $A_{\ell+1}, A_{\ell+2}, \ldots, A_m$ such that $F^{(n)}_{A_1A_2 \ldots A_m} = F^{(s)}_{A_1A_2 \ldots A_\ell} \circ F^{(n-1-s)}_{A_{\ell+1}A_{\ell+2} \ldots A_m}$ is given by

$$F^{(n)}_{A_1A_2 \ldots A_m}(X) = F^{(s)}_{A_1A_2 \ldots A_\ell}(X) \circ F^{(n-1-s)}_{A_{\ell+1}A_{\ell+2} \ldots A_m}(X) \; ,$$

where $\circ$ is one of $\oplus$, $\ominus$, $\otimes$, $\oplus$ and X is such an element of $\mathcal{I}$ that the right side is well-defined. It is clear that if $\mathcal{D}_1$ and $\mathcal{D}_2$ are the respective domains of $F^{(s)}_{A_1A_2 \ldots A_\ell}$ and $F^{(n-1-s)}_{A_{\ell+1}A_{\ell+2} \ldots A_m}$, then the domain $\mathcal{D}$ of $F^{(n)}_{A_1A_2 \ldots A_m}$ is given by

$$\mathcal{D} = \begin{cases} \mathcal{D}_1 \cap \mathcal{D}_2 \cap \left(F^{(n-1-s)}_{A_{\ell+1}A_{\ell+2} \ldots A_m}\right)^{-1}(\mathcal{J}) & \text{if } \circ \text{ is } \oplus \, , \\ \mathcal{D}_1 \cap \mathcal{D}_2 & \text{if otherwise.} \end{cases}$$

Let F be an arithmetic function of parameters $A_1, A_2, \ldots, A_m$ and domain $\mathcal{D}(F)$. We may write

$$F(A_1, A_2, \ldots, A_m ; X)$$

instead of $F(X)$ and consider F as a function of $\mathcal{J}^m \times \mathcal{D}(F)$ into $\mathcal{J}$. Notice that $\mathcal{D}(F)$ depends on $A_1, A_2, \ldots, A_m$.

(6-1) If $F(B_1, B_2, \ldots, B_m ; Y)$ is defined, then for any $C_1 \in \mathcal{J}_{B_1}$, $C_2 \in \mathcal{J}_{B_2}$, $\ldots$, $C_m \in \mathcal{J}_{B_m}$, $Z \in \mathcal{J}_Y$, $F(C_1, C_2, \ldots, C_m ; Z)$ is defined and is contained in $F(B_1, B_2, \ldots, B_m ; Y)$.

Proof. If F is of order 0 , our assertion is trivial. Hence our assertion holds for arithmetic functions of order 0 .

Now we proceed by induction. Let n be a positive integer and assume our assertion for all arithmetic functions of order $< n$. By definition every arithmetic function F of order n with parameters $A_1, A_2, \ldots, A_m$ is given by

$$F(A_1, A_2, \ldots, A_m ; X) = F_1(A_1, \ldots, A_\ell ; X) \circ F_2(A_{\ell+1}, \ldots, A_m ; X) ,$$

where F_1 and F_2 are arithmetic functions of order $< n$ and $\circ$ is one of $\oplus$, $\ominus$, $\otimes$, $\oplus$.

If $F(B_1, B_2, \ldots, B_m ; Y)$ is defined, then $F_1(B_1, \ldots, B_\ell ; Y)$ and $F_2(B_{\ell+1}, \ldots, B_m ; Y)$ is in $\mathcal{G}$ when $\circ$ is $\oplus$. By the induction hypothesis, for any $C_1 \in \mathcal{J}_{B_1}$, $C_2 \in \mathcal{J}_{B_2}$, $\ldots$, $C_m \in \mathcal{J}_{B_m}$, $Z \in \mathcal{J}_Y$, $F_1(C_1, \ldots, C_\ell ; Z)$ and $F_2(C_{\ell+1}, \ldots, C_m ; Z)$ are defined and

$$F_1(C_1, \ldots, C_\ell ; Z) \subset F_1(B_1, \ldots, B_\ell ; Y) ,$$

$$F_2(C_{\ell+1}, \ldots, C_m ; Z) \subset F_2(B_{\ell+1}, \ldots, B_m ; Y) .$$

So $F_2(C_{\ell+1}, \ldots, C_m; Z) \in \mathcal{J}$ when $\circ$ is $\oplus$. Hence

$$F(C_1, C_2, \ldots, C_m; Z) = F_1(C_1, \ldots, C_\ell; Z) \circ F_2(C_{\ell+1}, \ldots, C_m; Z)$$

is defined and is contained in $F(B_1, B_2, \ldots, B_m; Y)$. q.e.d.

(6-2) If $F(B_1, \ldots, B_m; Y)$ is defined, then there is a number $k > 0$ such that whenever $A_1, A_1' \in \mathcal{A}_{B_1}, \ldots, A_m, A_m' \in \mathcal{A}_{B_m}, X, X' \in \mathcal{A}_Y$, we have

$$P\Big(F(A_1, \ldots, A_m; X), F(A_1', \ldots, A_m'; X')\Big) \leqslant k\Big(P(A_1, A_1') + \ldots + P(A_m, A_m') + P(X, X')\Big) .$$

Hence F is uniformly continuous on $\mathcal{A}_{B_1} \times \ldots \times \mathcal{A}_{B_m} \times \mathcal{A}_Y$.

Proof. When F is of order o it is evident that the inequality holds with $k = 1$. Therefore we may proceed by induction and assume the inequality for arithmetic functions of order $< n$.

Every arithmetic function F of order n is given by

$$F(A_1, \ldots, A_m; X) = F_1(A_1, \ldots, A_\ell; X) \circ F_2(A_{\ell+1}, \ldots, A_m; X) ,$$

where F_1 and F_2 are arithmetic functions of order $< n$ and $\circ$ is one of $\oplus$, $\ominus$, $\otimes$, $\oplus$. By the induction hypothesis, there exist positive numbers k_1 and k_2 such that whenever $A_1, A_1' \in \mathcal{A}_{B_1}, \ldots, A_m, A_m' \in \mathcal{A}_{B_m}$, $X, X' \in \mathcal{A}_Y$, we have

$$P\Big(F_1(A_1, \ldots, A_\ell; X), F_1(A_1', \ldots, A_\ell'; X')\Big) \leqslant k_1\Big(P(A_1, A_1') + \ldots + P(A_\ell, A_\ell') + P(X, X')\Big)$$

$$P\Big(F_2(A_{\ell+1}, \ldots, A_m; X), F_2(A_{\ell+1}', \ldots, A_m'; X')\Big) \leqslant k_2\Big(P(A_{\ell+1}, A_{\ell+1}) + \ldots + P(A_m, A_m') + P(X, X')\Big) .$$

By (2-4), (3-6), (4-5), (5-3), there exists a positive number k for which our desired inequality holds. In fact, we can let

$$k = \begin{cases} 2(k_1 + k_2)\left(\gamma(J^{-1}) + \gamma(J^{-1})^2\,\gamma(I)\right) & \text{if } \circ \text{ is } \oplus, \\ 2(k_1 + k_2)\left(\gamma(I) + \gamma(J) + 1\right) & \text{if otherwise,} \end{cases}$$

where

$$I = F_1(B_1, \ldots, B_\ell ; Y) , \quad J = F_2(B_{\ell+1}, \ldots, B_m ; Y) .$$

q.e.d.

(6-3) If $F(B_1, \ldots, B_m; Y)$ is defined, then there is a $\delta > 0$ such that whenever $A_1, \ldots, A_m, X \in \mathcal{A}$ with

$$P(A_1,B_1) < \delta , \ \ldots, \ P(A_m,B_m) < \delta , \ P(X,Y) < \delta ,$$

$F(A_1, \ldots, A_m ; X)$ is defined and is continuous.

Proof. If F is of order 0, then our assertion is trivial. Therefore, we may proceed by induction and assume our assertion for arithmetic functions of order $< n$, $n > 0$.

Every arithmetic function F of order n is given by

$$F(A_1, \ldots, A_m ; X) = F_1(A_1, \ldots, A_\ell ; X) \circ F_2(A_{\ell+1}, \ldots, A_m ; X) ,$$

where F_1 and F_2 are arithmetic functions of order $< n$ and $\circ$ is one of $\oplus , \ominus , \otimes , \oplus$.

By the induction hypothesis there is a positive number δ such that whenever $A_1, \ldots, A_m, X \in \mathcal{A}$ with $P(A_1,B_1) < \delta , \ldots, P(A_m,B_m) < \delta , P(X,Y) < \delta$, both $F_1(A_1, \ldots, A_\ell ; X)$ and $F_2(A_{\ell+1}, \ldots, A_m ; X)$ are defined and continuous. If $F_2(B_{\ell+1}, \ldots, B_m; Y) \in \mathcal{J}$, we may choose δ so small that $F_2(A_{\ell+1}, \ldots, A_m ; X) \in \mathcal{J}$. Hence $F(A_1, \ldots, A_m ; X)$ is defined and is continuous.

q.e.d.

It follows from (6-1), (6-2) and (6-3) that

(6-4) Let F be an arithmetic function with m parameters and let

$$U = \{(A_1, \ldots, A_m; X) \in \mathcal{J}^{m+1} \mid F(A_1, \ldots, A_m; X) \text{ defined}\} .$$

Then U is open in $\mathcal{J}^{m+1}$ and F is a continuous function of U into $\mathcal{J}$. Moreover, for every $(B_1, \ldots, B_m; Y) \in U$, $\mathcal{J}_{B_1} \times \ldots \times \mathcal{J}_{B_m} \times \mathcal{J}_Y \subset U$ and F is uniformly continuous on $\mathcal{J}_{B_1} \times \ldots \times \mathcal{J}_{B_m} \times \mathcal{J}_Y$. Furthermore, $(A_1, \ldots, A_m; X) \in \mathcal{J}_{B_1} \times \ldots \times \mathcal{J}_{B_m} \times \mathcal{J}_Y$ implies

$$F(A_1, \ldots, A_m; X) \subset F(B_1, \ldots, B_m; Y) .$$

When we write $F(X)$ in place of $F(A_1, \ldots, A_m; X)$ it is understood that parameters $A_1, \ldots, A_m$ are fixed. Since

$$\mathcal{D}(F) = \{X \in \mathcal{J} \mid F(X) \text{ defined}\} ,$$

it follows from (6-4) and (6-2) that

(6-5) For every arithmetic function F, $\mathcal{D}(F)$ is open in $\mathcal{J}$ and $F: \mathcal{D}(F) \rightarrow \mathcal{J}$ is continuous. Let $I \in \mathcal{D}(F)$. Then $\mathcal{J}_I \subset \mathcal{D}(F)$ and $X \in \mathcal{J}_I$ implies $F(X) \subset F(I)$. Moreover, there is a positive number k such that whenever $X, X' \in \mathcal{J}_I$,

$$P\Big(F(X),\ F(X')\Big) \leqslant k P(X, X') .$$

7. RELATION BETWEEN ARITHMETIC FUNCTIONS AND RATIONAL FUNCTIONS.

In the construction of arithmetic functions, if we replace $\mathcal{J}$ by R and replace $\oplus$, $\ominus$, $\otimes$, $\oplus$ by corresponding operations on R, then we obtain rational functions in place of arithmetic functions. Therefore, we can establish a relation between arithmetic functions and rational function.

An arithmetic function is called special if all of its parameters belong to $p(R)$.

(7-1) If F is a special arithmetic function of domain $\mathcal{D}(F)$, then $p^{-1}\left(\mathcal{D}(F)\right)$ is open in R and for every $x \in p^{-1}\left(\mathcal{D}(F)\right)$, $Fp(x) \in p(R)$. Moreover, there is a unique rational function f whose domain contains $p^{-1}\left(\mathcal{D}(F)\right)$ such that $pf = Fp$ or $f = p^{-1}Fp$.

$$\begin{array}{ccc} R & \xrightarrow{\ p\ } & \mathcal{I} \\ \uparrow f & & \uparrow F \\ p^{-1}\left(\mathcal{D}(F)\right) & \xrightarrow{\ p\ } & \mathcal{D}(F) \end{array}$$

As the converse of (7-1), we have

(7-2) Given any rational function f of domain $D(f)$ there is a special arithmetic function F of domain $\mathcal{D}(F) \supset p\left(D(f)\right)$ such that $f = p^{-1}Fp$.

Remark. It is possible to have two distinct special arithmetic functions F_1 and F_2 such that $p^{-1}F_1p = p^{-1}F_2p$. For example,

$$F_1(X) = (X \otimes X) \oplus X,$$
$$F_2(X) = X \otimes \left(X \oplus p(1)\right), \qquad X \in \mathcal{I},$$

give two arithmetic functions F_1 and F_2 of domain $\mathcal{I}$. It it clear that

$$(p^{-1}F_1p)(x) = x^2 + x = x(x+1) = (p^{-1}F_2p)(x).$$

Since

$$F_1\left([-1, 0]\right) = [-1, 1], \quad F_2\left([-1, 0]\right) = [-1, 0],$$

F_1 and F_2 are distinct.

Let F be an arithmetic function with parameters $A_1, \ldots, A_m$. Let G be the arithmetic function with $B_1, \ldots, B_m$ in place of $A_1, \ldots, A_m$ respectively; that means, it is given by

$$G(X) = F(B_1, \ldots, B_m; X) .$$

If $B_1 \subset A_1, \ldots, B_m \subset A_m$, then, by (6-4), $\mathcal{D}(F) \subset \mathcal{D}(G)$ and for every $X \in \mathcal{D}(F)$, $G(X) \subset F(X)$. Hence the relation that $B_1 \subset A_1, \ldots, B_m \subset A_m$ will be written

$$G \subset F .$$

Let f be a rational function and let F be an arithmetic function. If there is a special arithmetic function $G \subset F$ with $f = p^{-1}Gp$, we say that f is an associated rational function of F or that F is an associated arithmetic function of f. In particular, if $G = F$ and then $f = p^{-1}Fp$, we say that F is an associated special arithmetic function of f and f is the associated rational function of F.

8. FIRST APPROXIMATION THEOREM

Let $X \in \mathcal{A}$. By a subdivision of X we mean

$$\xi = \left\{\xi_1, \xi_2, \ldots, \xi_r\right\}$$

such that

$$\alpha(X) = \alpha(\xi_1) < \beta(\xi_1) = \alpha(\xi_2) < \beta(\xi_2) = \alpha(\xi_3) <$$
$$\ldots < \beta(\xi_{r-1}) = \alpha(\xi_r) < \beta(\xi_r) = \beta(X) .$$

For every subdivision $\xi = \left\{\xi_1, \xi_2, \ldots, \xi_r\right\}$ we let

$$\sigma(\xi) = \max\left\{\sigma(\xi_1), \sigma(\xi_2), \ldots, \sigma(\xi_r)\right\} .$$

Let f be a rational function of domain $D(f)$; let G be an associated special arithmetic function of f and let F be an associated arithmetic function of f with $F \supset G$ (see § 7).

Let I be an element of $\mathcal{J}$ contained in $D(f)$. Then for every $x \in \mathcal{J}$, $G p(x) = p f(x)$ is well-defined. Since, by (6-5), the domain $\mathcal{D}(G)$ of G is open, there is, for every $x \in I$, a positive number r_x such that whenever $Y \in \mathcal{J}$ with $P(p(x), Y) < r_x$, $G(Y)$ is well-defined. Let

$$I_x = [x - r_x/2 , x + r_x/2] .$$

Then $G(I_x)$ is well-defined.

Since I is compact, there exist a finite number of points of I , say $x_1, x_2, \ldots, x_t$, such that I is contained in the union of the interior

$$Q_i = \left(x_i - r_{x_i}/2 , x_i + r_{x_i}/2\right)$$

of I_{x_i} , $i = 1, \ldots, r^t$. We abbreviate I_{x_i} by I_i .

Let $B_1, \ldots, B_m$ be the parameters of G and let $A_1, \ldots, A_m$ be the parameters of F . By definition,

$$B_1, \ldots, B_m \in p(R) ;$$

$$B_1 \subset A_1, \ldots, B_m \subset A_m ;$$

$$G(Y) = F(B_1, \ldots, B_m; Y) , \qquad Y \in \mathcal{D}(G) ;$$

$$F(Y) = F(A_1, \ldots, A_m; Y) , \qquad Y \in \mathcal{D}(F) .$$

By (6-3) there is, for every $i = 1, \ldots, t$, a $\delta_i > 0$ such that whenever $\sigma(A_1) < \delta_i, \ldots, \sigma(A_m) < \delta_i$, $F(I_i)$ is defined. Let

$$3\delta = \min\left\{\delta_1, \ldots, \delta_t\right\} .$$

Then whenever $\sigma(A_1) < 3\delta$, ..., $\sigma(A_m) < 3\delta$, $F(I_i)$ is defined for all i .

By a well-known theorem on compact metric spaces there exists a $\delta' > 0$ such that whenever $Y \in \mathcal{J}_I$ with $\sigma(Y) < \delta'$, Y is contained in one of $I_1, \ldots, I_t$ so that $F(Y)$ is defined by (6-5).

Let $X \in \mathcal{J}_I$ and let $\xi = \{\xi_1, \ldots, \xi_r\}$ be a subdivision of X with $\sigma(\xi) < \delta'$. Then

$$F(X,\xi) = F(\xi_1) \cup \ldots \cup F(\xi_r) \ ,$$

$$\Sigma(F,X,\xi) = \Big(F(\xi_1) \otimes p\sigma(\xi_1)\Big) \oplus \ldots \oplus \Big(F(\xi_r) \otimes p\sigma(\xi_r)\Big)$$

are well-defined when the parameters $A_1, \ldots, A_m$ satisfy

$$\sigma(A_1) < 3\delta \ , \ \ldots, \ \sigma(A_m) < 3\delta .$$

Clearly $\Sigma(F,X,\xi) \in \mathcal{J}$. Since for every $j = 2, \ldots, r$,

$$F(\xi_{j-1}) \cap F(\xi_j) \subset F\Big(p\alpha(\xi_j)\Big) \neq \phi \ ,$$

it follows that $F(X,\xi) \in \mathcal{J}$.

By (6-2), there is, for every $i = 1, \ldots, t$, a positive number k_i such that for any $Y, Y' \in \mathcal{J}_{I_i}$,

$$P\Big(G(Y),\ G(Y')\Big) \leqslant k_i P(Y,Y') \ ,$$

$$P\Big(F(Y),\ G(Y)\Big) \leqslant k_i\Big(\sigma(A_1) + \ldots + \sigma(A_m)\Big)$$

hold where the parameters satisfy

$$\sigma(A_1) < \delta, \ \ldots, \ \sigma(A_m) < \delta \ ,$$

and k_i is <u>independent of</u> the parameters.

Let $k = \max\left\{k_1, \ldots, k_t\right\}$.

Then for any $Y, Y' \in \mathscr{I}_I$ with $\sigma(Y) < \delta'$ and $\sigma(Y') < \delta'$

$$P\Big(F(Y), G(Y')\Big) \leqslant k\Big(P(Y,Y') + \sigma(A_1) + \ldots + \sigma(A_m)\Big) .$$

In fact, there is a finite sequence

$$Y_1 = Y, Y_2, \ldots, Y_s = Y'$$

in $\mathscr{I}_I$ such that

$$\sigma(Y_2) = \ldots = \sigma(Y_s) = \sigma(Y') ,$$

$$\alpha(Y_2) < \ldots < \alpha(Y_s) = \alpha(Y')$$

and Y_{j-1} and Y_j are contained in the same I_i for some i . Hence

$$\begin{aligned} kP(Y, Y') &= k\Big(P(Y_1,Y_2) + \ldots + P(Y_{s-1},Y_s)\Big) \\ &\geqslant P\Big(G(Y_1), G(Y_2)\Big) + \ldots + P\Big(G(Y_{s-1}), G(Y_s)\Big) \\ &\geqslant P\Big(G(Y), G(Y')\Big) . \end{aligned}$$

Moreover

$$k\Big(\sigma(A\ \ ) + \ldots + \sigma(A_m)\Big) \geqslant P\Big(F(Y), G(Y)\Big) .$$

Our assertion thus follows.

Now we are ready to prove

Theorem 1. Let f be a rational function of domain $D(f)$, let G be an associated special arithmetic function of f and let I be an element of $\mathscr{I}$ contained in $D(f)$. Then there are positive numbers δ, δ', k such that whenever F is an associated arithmetic function of f with *parameters* $A_1, \ldots, A_m$ *such that* $F \supset G$ *and* $\sigma(A_1) < \delta, \ldots, \sigma(A_m) < \delta$, X *is an element of* $\mathcal{I}_I$ *and* $\xi = \{\xi_1, \ldots, \xi_r\}$ *is a subdivision of* X *with* $\sigma(\xi) < \delta'$,

$$F(X,\xi) = F(\xi_1) \cup \ldots \cup F(\xi_r)$$

is defined and satisfies

$$f(X) \subset F(X,\xi) \subset f(X) \oplus [-\kappa, \kappa] ,$$

where

$$f(X) = \{f(x) \mid x \in X\} ,$$

$$\kappa = k\Big(\sigma(\xi) + \sigma(A_1) + \ldots + \sigma(A_m)\Big) .$$

Proof. Let δ, δ', k be chosen as above. Let F be an associated arithmetic function of f with parameters $A_1, \ldots, A_m$ such that $F \subset G$ and $\sigma(A_1) < \delta, \ldots, \sigma(A_m) < \delta$. Let $X \in \mathcal{I}_I$ and let ξ be a subdivision of X with $\sigma(\xi) < \delta'$. We have shown that $F(X,\xi)$ is defined.

For every $y \in f(X)$ there is an $x \in X$ with $f(x) = y$. Let $x \in \xi_j$. Then, by (6-5),

$$y = f(x) \in F\,p(x) \subset F(\xi_j) \subset F(X,\xi) .$$

Hence $f(X) \subset F(X,\xi)$.

For every $y \in F(X,\xi)$, there is a ξ_j with $y \in F(\xi_j)$. Let $x \in \xi_j$. Then $P\Big(\xi_j , p(x)\Big) \leqslant \sigma(\xi_j) < \sigma(\xi)$. It follows that

$$P\Big(F(\xi_j), Gp(x)\Big) \leqslant k\Big(\sigma(\xi) + \sigma(A_1) + \ldots + \sigma(A_m)\Big)$$

or

$$P\Big(F(\xi_j) , pf(x)\Big) \leqslant \kappa .$$

Therefore

$$y \in F(\xi_j) \subset pf(x) \oplus [-\kappa, \kappa]$$
$$\subset f(X) \oplus [-\kappa, \kappa]$$

Hence $F(X,\xi) \subset f(X) \oplus [-\kappa, \kappa]$. q.e.d.

Corollary 1. Let f be a rational function of domain $D(f)$, let G be an associated special arithmetic function of f and let X be an element of $\mathscr{I}$ contained in $D(f)$. Let F be an associated arithmetic function of f with parameters $A_1, \ldots, A_m$ such that $F \supset G$. Then whenever ξ is a subdivision of X with small $\sigma(\xi)$ and $\sigma(A_1), \ldots, \sigma(A_m)$ are small, $F(X,\xi)$ is defined. Moreover, as $\sigma(\xi) + \sigma(A_1) + \ldots + \sigma(A_m) \longrightarrow 0$,

$$\lim F(X,\xi) = f(X) ,$$

that means,

$$\lim P\Big(F(X,\xi) , f(X)\Big) = 0 .$$

9. SECOND APPROXIMATION THEOREM

Theorem 2. Let f be a rational function of domain $D(f)$, let G be an associated special arithmetic function of f and let I be an element of $\mathscr{I}$ contained in $D(f)$. Then there are positive numbers δ, δ', k such that whenever F is an associated arithmetic function of F with parameters $A_1, \ldots, A_m$ such that $F \supset G$ and $\sigma(A_1) < \delta, \ldots, \sigma(A_m) < \delta$, $X = [a,b]$ is an element of $\mathscr{I}_I$ and ξ is a subdivision of X ,

$$\Sigma(F,X,\xi) = \Big(F(\xi_1) \otimes p\sigma(\xi_1)\Big) \oplus \cdots \oplus \Big(F(\xi_r) \otimes p\sigma(\xi_r)\Big)$$

is defined and satisfies

$$p\left(\int_a^b f(x)dx\right) \subset \Sigma(F,X,\xi) \subset p\left(\int_a^b f(x)dx\right) + \Big[-\kappa\sigma(X) , \kappa\sigma(X)\Big] ,$$

where

$$\kappa = k\Big(\sigma(\xi) + \sigma(A_1) + \dots + \sigma(A_m)\Big) .$$

Proof. Let δ, δ', k be as before. Let F be an associated arithmetic function of F with parameters $A_1, \dots, A_m$ such that $F \supset G$ and $\sigma(A_1) < \delta, \dots, \sigma(A_m) < \delta$. Let $X = [a,b]$ be an element of $\mathcal{J}_I$ and let $\xi = \{\xi_1, \dots, \xi_r\}$ be a subdivision of X with $\sigma(\xi) < \delta'$. We have shown that $\Sigma(F,X,\xi)$ is defined.

Let

$$m_j = \inf_{x \in \xi_j} f(x) \quad , \qquad M_j = \sup_{x \in \xi_j} f(x) .$$

Then

$$\Sigma_{j=1}^{r} m_j \, \sigma(\xi_j) \leqslant \int_a^b f(x)dx \leqslant \Sigma_{j=1}^{r} M_j \, \sigma(\xi_j) .$$

Since ξ_j is compact, there is a point x_j of ξ_j with $f(x_j) = m_j$. Then

$$m_j = f(x_j) \in F\,p\,(x_j) \subset F\,(\xi_j)$$

so that

$$m_j \sigma(\xi_j) \in F(\xi_j) \otimes p\sigma(\xi_j) .$$

Hence

$$\Sigma_{j=1}^{r} m_j \, \sigma(\xi_j) \in \Sigma(F,X,\xi) .$$

Similarly we can show that

$$\Sigma_{j=1}^{r} M_j \, \sigma(\xi_j) \in \Sigma(F,X,\xi) .$$

Since $\Sigma(F,X,\xi) \in \mathcal{A}$, it follows that

$$p\left(\int_a^b f(x)dx\right) \subset \Sigma(F,X,\xi) \quad .$$

Let x_j be as above. Since

$$P\left(\xi_j, p(x_j)\right) \leqslant \sigma(\xi_j) \leqslant \sigma(\xi) \quad ,$$

it follows that

$$P\left(F(\xi_j), Gp(x_j)\right) \leqslant k\left(\sigma(\xi) + \sigma(A_1) + \dots + \sigma(A_m)\right) = \kappa$$

so that

$$F(\xi_j) \subset p(m_j) \oplus [-\kappa, \kappa]$$

Therefore

$$F(\xi_j) \otimes p\sigma(\xi_j) \subset p\left(m_j\ \sigma(\xi_j)\right) \oplus \left[-\kappa\ \sigma(\xi_j), \kappa\ \sigma(\xi_j)\right] \quad .$$

Hence

$$\Sigma(F,X,\xi) \subset p\left(\Sigma_{j=1}^{r}\ m_j\ \sigma(\xi_j)\right) \oplus \left[-\kappa\ \sigma(X), \kappa\ \sigma(X)\right] \quad .$$

Similarly we can prove that

$$\Sigma(F,X,\xi) \subset p\left(\Sigma_{j=1}^{r}\ m_j\ \sigma(\xi_j)\right) \oplus \left[-\kappa\ \sigma(X), \kappa\ \sigma(X)\right] \quad .$$

Since $\Sigma(F,X,\xi) \in \mathcal{A}$, it follows that

$$\Sigma(F,X,\xi) \subset p\left(\int_a^b f(x)dx\right) \oplus \left[-\kappa\ \sigma(X), \kappa\ \sigma(X)\right] \quad .$$

q.e.d.

Corollary. Let f be a rational function of domain $D(f)$, let G be an associated special arithmetic function of f and let $X = [a,b]$ be an element of $\mathscr{J}$ contained in $D(f)$. Let F be an associated arithmetic function of f with parameters $A_1, \ldots, A_m$ such that $F \supset G$. Then whenever ξ is a subdivision of X with small $\sigma(\xi)$ and $\sigma(A_1), \ldots, \sigma(A_m)$ are small, $\Sigma(F,X,\xi)$ are defined. Moreover, as $\sigma(\xi) + \sigma(A_1) + \ldots + \sigma(A_m) \longrightarrow 0$,

$$\lim \; \Sigma(F,X,\xi) = \int_a^b f(x)dx \quad ,$$

that means

$$\lim \; P\left(\Sigma(F,X,\xi),\; p\left(\int_a^b f(x)dx\right)\right) = 0 \quad .$$

10. APPROXIMATION OF A CONTINUOUS FUNCTION BY ARITHMETIC FUNCTIONS

Let I be an element of $\mathscr{J}$ and let $f = I \longrightarrow R$ be a continuous function. Let

$$\{F_n\} = \{F_1,\; F_2,\; \ldots\}$$

be a sequence of arithmetic functions such that $p(I)$ is contained in the domain $\mathscr{D}(F_n)$ for all n . If for every $x \in I$,

$$F_1 p(x) \supset F_2 p(x) \supset \ldots$$

and

$$\bigcap_{n=1}^{\infty} F_n p(x) = pf(x) \quad ,$$

we say that $\{F_n\}$ <u>converges to</u> f . In symbols,

$$\lim_{n \to \infty} F_n = f \quad .$$

(10-1) Let f be a rational function of domain $D(f)$ and let I be an element of $\mathscr{J}$ contained in $D(f)$. Let F be an associated arithmetic function of f with parameters $A_1, \ldots, A_m$. Then for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $\sigma(A_1) < \delta, \ldots, \sigma(A_m) < \delta$, then for every $x \in I$, $Fp(x)$ is defined and $\sigma\big(Fp(x)\big) < \varepsilon$.

Proof. Let G be the associated special arithmetic function of f with $F \supset G$ and let $B_1, \ldots, B_m$ be the parameters of G. By (6-3) there is, for every $x \in I$, a $\delta_x > 0$ such that if $P(A_1,B_1) < \delta_x, \ldots,$ $P(A_m,B_m) < \delta_x$ and X is an element of $\mathcal{I}_I$ with $P\big(X,p(x)\big) < \delta_x$, then $F(X)$ is defined and $P\big(F(X), Gp(x)\big) < \varepsilon/2$. Since I is compact there exist a finite number of points of I, say $x_1, \ldots, x_t$, such that the union of

$$\left(x_j - \delta_{x_j}, x_j + \delta_{x_j}\right), \qquad j = 1, \ldots, t,$$

contains I. Let

$$\delta = \min\left\{\delta_{x_1}, \ldots, \delta_{x_t}\right\}.$$

It follows that if $P(A_1,B_1) < \delta \ \ldots, P(A_m,B_m) < \delta$, then for every $x \in I$,

$$\sigma\big(Fp(x)\big) < \varepsilon .$$

In fact, there is an x_j with $P\big(p(x), p(x_j)\big) < \delta_j$ so that $P\big(Fp(x), pf(x)\big) < \varepsilon/2$. Hence

$$\sigma\big(Fp(x)\big) < \varepsilon . \qquad \text{q.e.d.}$$

Let f be a rational function of domain $D(f)$ and let I be an element of $\mathcal{I}$ contained in $D(f)$. By (10-1) we can easily construct a sequence of arithmetic functions

$$F_1 \supset F_2 \supset \ldots$$

such that for every $x \in I$,

$$\sigma\left(F_n p(x)\right) < 1/n \quad , \qquad n = 1, 2, \ldots \quad .$$

Hence

$$\lim_{n \to \infty} F_n = f \quad .$$

This result can be extended as follows.

(10-2) For every continuous function $f : I \to R$, $I \in \mathcal{J}$, there is a sequence of arithmetic functions

$$F_n \, , \qquad n = 1, 2, \ldots$$

with

$$\lim_{n \to \infty} F_n = f \quad .$$

Proof. It is well-known that every continuous function can be approximated by polynomials. For every integer $n > 1$ we let f_n be a polynomial such that for every $x \in I$, $\rho\left(f_n(x), f(x)\right) < 1/(6 \cdot 3^n)$. By (10-1), there is an arithmetic function G_n such that for every $x \in I$, $G_n p(x)$ is defined and

$$P\left(G_n p(x), pf_n(x)\right) \leqslant 1/(6 \cdot 3^n)$$

Let F_n be the arithmetic function such that

$$F_n(X) = G_n(X) \oplus \left[-4/(6 \cdot 3^n), 4/(6 \cdot 3^n)\right] \quad , \qquad X \in \mathcal{D}(G_n) \quad .$$

Then for every $x \in I$, $F_n p(x)$ is defined. Since

$$P\Big(G_n p(x),\ pf(x)\Big) \leqslant P\Big(G_n p(x),\ pf_n(x)\Big) + P\Big(pf_n(x),\ pf(x)\Big)$$
$$\leqslant 1/(6\cdot 3^n) + 1/(6\cdot 3^n) \doteq 2/(6\cdot 3^n) \quad ,$$

it follows that

$$pf(x) \subset G_n p(x) \oplus \Big[-2/(6\cdot 3^n),\ 2/(6\cdot 3^n)\Big] \quad ,$$
$$G_n p(x) \subset pf(x) \oplus \Big[-2/(6\cdot 3^n),\ 2/(6\cdot 3^n)\Big] \quad .$$

Hence

$$pf(x) \oplus \Big[-2/(6\cdot 3^n),\ 2/(6\cdot 3^n)\Big] \subset F_n p(x)$$
$$\subset pf(x) \oplus \Big[-1/3^n,\ 1/3^n\Big]$$

Consequently

$$F_{n+1}p(x) \subset pf(x) \oplus \Big[-1/3^{n+1},\ 1/3^{n+1}\Big] \subset F_n p(x) \quad .$$

Since $\sigma\Big(F_n p(x)\Big) < 2/3^n$ and $\lim\limits_{n\to\infty} 2/3^n = 0$, it follows that $\lim F_n = f$.

q.e.d.

Let I be an element of $\mathcal{J}$ and let

$$\xi = \Big\{\xi_1,\ \ldots,\ \xi_r\Big\}, \qquad\qquad \eta = \Big\{\eta_1\ ,\ \ldots,\ \eta_s\Big\}$$

be subdivisions of I . If there exist integers $1 \leqslant j(1) < j(2) < \ldots < j(r) = s$ such that for every $i = 1,\ \ldots,\ r,$ $\Big\{\eta_{j(i-1)+1}\ ,\ \ldots,\ \eta_{j(i)}\Big\}$ is a subdivision of ξ_i , we write $\xi \prec \eta$ and call η a <u>refinement</u> of ξ .

Let $I \in \mathcal{J}$ and let F be an arithmetic function with $\mathcal{D}(F) \supset p(I)$. As before, there is a positive number δ such that whenever $X \in \mathcal{J}_I$ with

$\sigma(X) < \delta$, $F(X)$ is defined. Hence if ξ is a subdivision of I with $\sigma(\xi) < \delta$, both $F(I,\xi)$ and $\Sigma(F,I,\xi)$ are defined.

(10-3) Let F be an arithmetic function and let $\xi = \{\xi_1, \ldots, \xi_r\}$ be a subdivision of $I \in \mathcal{J}$ such that $F(\xi_1), \ldots, F(\xi_r)$ are defined. Then for every refinement $\eta = \{\eta_1, \ldots, \eta_s\}$ of ξ , $F(\eta_1), \ldots, F(\eta_s)$ are defined so that $F(I,\eta)$ and $\Sigma(F,I,\eta)$ are defined. Moreover,

$$F(I,\eta) \subset F(I,\xi) , \qquad \Sigma(F,I,\eta) \subset \Sigma(F,I,\xi) .$$

(10-4) Let $I \in \mathcal{J}$ and let F be an arithmetic function. Let $\xi = \{\xi_1, \ldots, \xi_r\}$ be a subdivision of I such that $F(\xi_1), \ldots, F(\xi_r)$ are defined, and let

$$\xi = \xi^{(1)} \prec \xi^{(2)} \prec \ldots$$

be a sequence of subdivisions of I . Then $F\left(I,\xi^{(n)}\right)$ and $\Sigma\left(F,I,\xi^{(n)}\right)$ are defined for all n and

$$F\left(I,\xi^{(1)}\right) \supset F\left(I,\xi^{(2)}\right) \supset \ldots ,$$

$$\Sigma\left(F,I,\xi^{(1)}\right) \supset \Sigma\left(F,I,\xi^{(2)}\right) \supset \ldots .$$

If $\lim_{n\to\infty} \sigma\left(\xi^{(n)}\right) = 0$, then $\bigcap_{n=1}^{\infty} F\left(I,\xi^{(n)}\right)$ and $\bigcap_{n=1}^{\infty} \Sigma\left(F,I,\xi^{(n)}\right)$ are independent of the choice of ξ and $\left\{\xi^{(n)}\right\}$.

Proof. Let $\eta = \{\eta_1, \ldots, \eta_s\}$ be a subdivision of I such that $F(\eta_1), \ldots, F(\eta_s)$ are defined and let $\zeta = \{\zeta_1, \ldots, \zeta_t\}$ be a refinement of η . By definition, there are integers $1 \leq j(1) < \ldots < j(s) = t$

such that for every $i = 1, \ldots, s$, $\left\{\zeta_{j(i-1)+1}, \ldots, \zeta_{j(i)}\right\}$ is a subdivision of η_i. It follows from (6-5) that

$$F\left(\zeta_{j(i-1)+1}\right) \subset F(\eta_i), \ldots, F\left(\zeta_{j(i)}\right) \subset F(\eta_i) \quad .$$

Hence

$$\begin{aligned} F(I,\zeta) &= F(\zeta_1) \cup \ldots \cup F(\zeta_t) \\ &\subset F(\eta_1) \cup \ldots \cup F(\eta_s) = F(I,\eta) \end{aligned}$$

As a consequence of this result we have

$$F\left(I,\xi^{(1)}\right) \supset F\left(I,\xi^{(2)}\right) \supset \ldots$$

Since

$$\begin{aligned} &\alpha\left(\left(F\left(\zeta_{j(i-1)+1}\right) \otimes p\sigma\left(\zeta_{j(i-1)+1}\right)\right) \oplus \cdots \oplus \left(F\left(\zeta_{j(i)}\right) \otimes p\sigma\left(\zeta_{j(i)}\right)\right)\right) \\ &= \alpha\left(F\left(\zeta_{j(i-1)+1}\right)\right) \sigma\left(\zeta_{j(i-1)+1}\right) + \cdots + \alpha\left(F\left(\zeta_{j(i)}\right)\right) \sigma\left(\zeta_{j(i)}\right) \\ &\geqslant \alpha\left(F(\eta_i)\right) \sigma\left(\zeta_{j(i-1)+1}\right) + \cdots + \alpha\left(F(\eta_i)\right) \sigma\left(\zeta_{j(i)}\right) \\ &= \alpha\left(F(\eta_i)\right) \sigma(\eta_i) = \alpha\left(F(\eta_i) \otimes p\sigma(\eta_i)\right) \end{aligned}$$

and similarly

$$\begin{aligned} &\beta\left(\left(F\left(\zeta_{j(i-1)+1}\right) \otimes p\sigma\left(\zeta_{j(i-1)+1}\right)\right) \oplus \cdots \oplus \left(F\left(\zeta_{j(i)}\right) \otimes p\sigma\left(\zeta_{j(i)}\right)\right)\right) \\ &\leqslant \beta\left(F(\eta_i) \otimes p\sigma(\eta_i)\right) \quad , \end{aligned}$$

it follows that

$$\left(F\left(\zeta_{j(i-1)+1}\right) \otimes p\sigma\left(\zeta_{j(i-1)+1}\right)\right) \oplus \cdots \oplus \left(F\left(\zeta_{j(i)}\right) \otimes p\sigma\left(\zeta_{j(i)}\right)\right)$$

$$\subset F(\eta_i) \otimes p\sigma(\eta_i) \quad .$$

Hence

$$\Sigma(F,I,\zeta) = \left(F(\zeta_1) \otimes p\sigma(\zeta_1)\right) \oplus \cdots \oplus \left(F(\zeta_t) \otimes p\sigma(\zeta_t)\right)$$

$$\subset \left(F(\eta_1) \otimes p\sigma(\eta_1)\right) \oplus \cdots \oplus \left(F(\eta_s) \otimes p\sigma(\eta_s)\right) = \Sigma(F,I,\eta) \quad .$$

From this result it follows that

$$\Sigma\left(F,I,\xi^{(1)}\right) \supset \Sigma\left(F,I,\xi^{(2)}\right) \supset \cdots$$

Now we assume $\lim_{n\to\infty} \sigma\left(\xi^{(n)}\right) = 0$. Let $\eta = \left\{\eta_1, \ldots, \eta_s\right\}$ be a subdivision of I such that $F(\eta_1), \ldots, F(\eta_s)$ are defined, and let

$$\eta = \eta^{(1)} \prec \eta^{(2)} \prec \cdots$$

be a sequence of subdivisions with $\lim_{n\to\infty} \sigma\left(\eta^{(n)}\right) = 0$.

Let $\varepsilon > 0$. For any $y \in \bigcap_{n=1}^{\infty} F\left(I,\xi^{(n)}\right)$ there is a sequence

$$\xi^{(1)}_{i(1)} \supset \xi^{(2)}_{i(2)} \supset \cdots .$$

in $\mathcal{J}$ such that for every $n = 1, 2, \ldots,$ $\xi^{(n)}_{i(n)} \in \xi^{(n)}$ and $F\left(\xi^{(n)}_{i(n)}\right) \ni y$. Since $\lim_{n\to\infty} \sigma\left(\xi^{(n)}\right) = 0$, $\bigcap_{n=1}^{\infty}\left(\xi^{(n)}_{i(n)}\right)$ contains a single point x . By (6-5) there is a $\delta > 0$ such that whenever $X \in \mathcal{J}_I$ with $P\left(X, p(x)\right) < \delta$,

$F(X)$ is defined and $P\left(F(X), Fp(x)\right) < \varepsilon$. Since $\lim_{n\to\infty} \sigma\left(\eta^{(n)}\right) = 0$, $\sigma\left(\eta^{(n)}\right) < \delta$ holds for all large n. Let $\eta_i^{(n)}$ be the element of $\eta^{(n)}$ containing x. Then $P\left(\eta_i^{(n)}, p(x)\right) < \delta$ so that $P\left(F\left(\eta_i^{(n)}\right) Fp(x)\right) < \varepsilon$. Hence for all large n

$$y \in Fp(x) \subset F\left(\eta_i^{(n)}\right) \oplus [-\varepsilon, \varepsilon] \subset F\left(I, \eta^{(n)}\right) \oplus [-\varepsilon, \varepsilon]$$

This proves that $y \in \bigcap_{n=1}^{\infty} F\left(I, \eta^{(n)}\right) \oplus [-\varepsilon, \varepsilon]$. Since y is an arbitrary point of $\bigcap_{n=1}^{\infty} F\left(I, \xi^{(n)}\right)$, it follows that

$$\bigcap_{n=1}^{\infty} F\left(I, \xi^{(n)}\right) \subset \bigcap_{n=1}^{\infty} F\left(I, \eta^{(n)}\right) \oplus [-\varepsilon, \varepsilon] \ .$$

Similarly,

$$\bigcap_{n=1}^{\infty} F\left(I, \eta^{(n)}\right) \subset \bigcap_{n=1}^{\infty} F\left(I, \xi^{(n)}\right) \oplus [-\varepsilon, \varepsilon] \ .$$

Applying (1-6), we have

$$\bigcap_{n=1}^{\infty} F\left(I, \xi^{(n)}\right) = \bigcap_{n=1}^{\infty} F\left(I, \eta^{(n)}\right) \ .$$

In order to prove that $\bigcap_{n=1}^{\infty} \Sigma\left(F, I, \xi^{(n)}\right) = \bigcap_{n=1}^{\infty} \Sigma\left(F, I, \eta^{(n)}\right)$, we may assume $\sigma(I) > 0$. It is sufficient to prove that for every integer $m \geqslant 1$ and every $\varepsilon > 0$,

$$\Sigma\left(F, I, \xi^{(m)}\right) \oplus [-\varepsilon, \varepsilon] \supset \Sigma\left(F, I, \eta^{(n)}\right)$$

holds for large n. In fact, if this is proved, then

$$\Sigma\left(F, I, \xi^{(m)}\right) \oplus [-\varepsilon, \varepsilon] \supset \bigcap_{n=1}^{\infty} \Sigma\left(F, I, \eta^{(n)}\right) \ .$$

Since ε is arbitrary, it follows that $\Sigma\left(F, I, \xi^{(m)}\right) \supset \bigcap_{n=1}^{\infty} \Sigma\left(F, I, \eta^{(n)}\right)$.

Since m is arbitrary, it follows that

$$\bigcap_{m=1}^{\infty} \Sigma\left(F, I, \xi^{(m)}\right) \supset \bigcap_{n=1}^{\infty} \Sigma\left(F, I, \eta^{(n)}\right) .$$

Similarly,

$$\bigcap_{n=1}^{\infty} \Sigma\left(F, I, \eta^{(n)}\right) \supset \bigcap_{m=1}^{\infty} \Sigma\left(F, I, \xi^{(m)}\right) .$$

Hence our assertion follows.

Now we let m be an arbitrary integer $\geqslant 1$ and let $\xi^{(m)} = \left\{A_1, \ldots, A_u\right\}$. Let ε be an arbitrary positive number < 1. By (6-3), there is, for every $x \in I$, a positive number r_x such that whenever $X \in \mathscr{A}$ with $P\left(X, p(x)\right) < r_x$, $F(X)$ is defined and $P\left(F(X), Fp(x)\right) < \varepsilon/u\left(\gamma\left(F\left(I, \xi^{(m)}\right)\right) + 1\right)$. Since I is compact, there is a $\delta > 0$ such that whenever $X \in \mathscr{A}_I$ with $\sigma(X) < \delta$, $P\left(X, p(x)\right) < r_x$ for some $x \in I$.

Since $\lim_{n \to \infty} \sigma\left(\eta^{(n)}\right) = 0$, there is an integer n_0 such that

$$\sigma\left(\eta^{(n)}\right) < \min\left(\delta, \varepsilon/u\left(\gamma\left(F\left(I, \xi^{(m)}\right)\right) + 1\right)\right)$$

for all integers $> n_0$. Let $n > n_0$ and let

$$\eta^{(n)} = \left\{B_1, \ldots, B_v\right\} .$$

For every $i = 1, \ldots, u$, there is a largest integer $j(i)$ with $\beta(A_i) \in B_{j(i)}$. Clearly

$$1 \leqslant j(1) \leqslant j(2) \leqslant \ldots \leqslant j(u) = v .$$

Since $P\left(B_{j(i)}, p\beta(A_i)\right) < \delta$, it follows that

$$P\left(F\left(B_{j(i)}\right), Fp\beta(A_i)\right) < \varepsilon/u\left(\gamma\left(F\left(I, \xi^{(m)}\right)\right) + 1\right) < 1 .$$

Therefore

$$F\left(B_{j(i)}\right) \subset F\, p\, \beta(A_i) \oplus [-1, 1] \subset F(A_i) \oplus [-1, 1]$$

and then

$$\gamma\left(F\left(B_{j(i)}\right) \otimes p\,\sigma\left(B_{j(i)}\right)\right) \leqslant \left(\gamma\left(F(A_i)\right) + 1\right) \sigma\left(B_{j(i)}\right)$$
$$\leqslant \left(\gamma\left(F\left(I, \xi^{(m)}\right)\right) + 1\right) \sigma\left(B_{j(i)}\right) .$$

Hence

$$\Sigma_{i=1}^{u}\, \gamma\left(F\left(B_{j(i)}\right) \otimes p\,\sigma\left(B_{j(i)}\right)\right) \leqslant \left(\gamma\left(F\left(I, \xi^{(m)}\right)\right) + 1\right) \Sigma_{i=1}^{m}\, \sigma\left(B_{j(i)}\right)$$
$$\leqslant \left(\gamma\left(F\left(I, \xi^{(m)}\right)\right) + 1\right) u\, \sigma\left(\eta^{(n)}\right) < \varepsilon$$

and consequently

$$\left(F\left(B_{j(1)}\right) \otimes p\,\sigma\left(B_{j(1)}\right)\right) \oplus \cdots \oplus \left(F\left(B_{j(u)}\right) \otimes p\,\sigma\left(B_{j(u)}\right)\right) \subset [-\varepsilon, \varepsilon]$$

Since, for every $k = j(i-1) + 1, \ldots, j(i) - 1$, $B_k \subset A_i$, it follows that

$$\left(F\left(B_{j(i-1)+1}\right) \otimes p\,\sigma\left(B_{j(i-1)+1}\right)\right) \oplus \cdots \oplus \left(F\left(B_{j(i)-1}\right) \otimes p\,\sigma\left(B_{j(i)-1}\right)\right)$$
$$\subset F(A_i) \otimes p\,\sigma(A_i) \quad , \qquad\qquad i = 1, \ldots, u \quad .$$

Hence, by adding these equations, we have

$$\Sigma\left(F, I, \eta^{(n)}\right) \subset \Sigma\left(F, I, \xi^{(m)}\right) \oplus [-\varepsilon, \varepsilon] \quad .$$

This completes our proof. q.e.d.

Let F be an arithmetic function with domain $\mathcal{D}(F)$. Let $\mathcal{J}^F$ be the subset of $\mathcal{J}$ consisting of all the elements X of $\mathcal{J}$ with $p(X) \subset \mathcal{D}(F)$. Then, by (10-4), we may define a function

$$\overline{F} : \mathcal{J}^F \longrightarrow \mathcal{J}$$

by

$$\overline{F}(X) = \bigcap_{n=1}^{\infty} F\left(X, \xi^{(n)}\right) ,$$

where $X \in \mathcal{J}^F$ and $\xi^{(1)} \prec \xi^{(2)} \prec \ldots$ is a sequence of subdivisions of X with $\lim_{n \to \infty} \sigma\left(\xi^{(n)}\right) = 0$.

(10-5) Let X and Y be elements of $\mathcal{J}^F$ with $\beta(X) = \alpha(Y)$. Then $X \cup Y \in \mathcal{J}^F$ and

$$\overline{F}(X \cup Y) = \overline{F}(X) \cup \overline{F}(Y) .$$

Proof. Since $\beta(X) = \alpha(Y)$, $X \cup Y \in \mathcal{J}$. Since $X \cup Y \in \mathcal{J}^F$, $p(X) \subset \mathcal{D}(F)$ and $p(Y) \subset \mathcal{D}(F)$. Hence $p(X \cup Y) \subset \mathcal{D}(F)$ and consequently $X \cup Y \in \mathcal{J}^F$.

If $Y \in p(R)$, then $\overline{F}(Y) = F(Y) \subset \overline{F}(X)$ so that our assertion is obvious. If $Y \notin p(R)$, then we may have a sequence of subdivisions

$$\xi^{(n)} = \left\{\xi_1^{(n)}, \ldots, \xi_{r(n)}^{(n)}\right\}$$

of $X \cup Y$ such that $\xi^{(1)} \prec \xi^{(2)} \prec \ldots$, $\lim_{n \to \infty} \sigma\left(\xi^{(n)}\right) = 0$ and for every integer n there is an integer $s(n)$ with $\beta\left(\xi_{s(n)}^{(n)}\right) = \beta(X)$. Therefore

$$\eta^{(n)} = \left\{\xi_1^{(n)}, \ldots, \xi_{s(n)}^{(n)}\right\}$$

is a sequence of subdivisions of X such that $\eta^{(1)} \prec \eta^{(2)} \prec \ldots$ and $\lim_{n \to \infty} \sigma\left(\eta^{(n)}\right) = 0$; and $\zeta^{(n)} = \left\{\xi_{s(n)+1}^{(n)}, \ldots, \xi_{r(n)}^{(n)}\right\}$ is a sequence of

subdivisions of Y such that $\zeta^{(1)} \prec \zeta^{(2)} \prec \ldots$ and $\lim_{n\to\infty} \sigma\left(\zeta^{(n)}\right) = 0$.
Since

$$F\left(X \cup Y, \xi^{(n)}\right) = F\left(X, \eta^{(n)}\right) \cup F\left(Y, \zeta^{(n)}\right),$$

it follows that

$$\overline{F}(X \cup Y) = \overline{F}(X) \cup \overline{F}(Y) \quad . \qquad \text{q.e.d.}$$

(10-6) $\overline{F} : \mathscr{S}^F \longrightarrow \mathscr{S}$ is continuous.

Proof. Let $Y \in \mathscr{S}^F$ and let $\varepsilon > 0$. Since $F\left(\alpha(Y)\right)$ is defined, there is a $\delta > 0$ such that whenever $Z \in \mathscr{S}$ with $\sigma(Z) < \delta$, $P\left(Z, p\alpha(Y)\right) < \delta$, $F(Z)$ is defined and $P\left(F(Z), Fp\alpha(Y)\right) < \varepsilon/4$. Since

$$F(Z) \supset \overline{F}(Z) \supset Fp\,\alpha(Y) = \overline{F}p\alpha(Y) \quad ,$$

we have

$$P\left(\overline{F}(Z), \overline{F}p\,\alpha(Y)\right) < \varepsilon/4 \quad .$$

Similarly there is a $\delta' > 0$ such that whenever $Z' \in \mathscr{S}$ with $P\left(Z', p\beta(Z)\right) < \delta'$, $F(Z')$ is defined and

$$P\left(\overline{F}(Z'), \overline{F}p\,\beta(Y)\right) < \varepsilon/4 \quad .$$

For every $X \in \mathscr{S}_F$ with $P(X, Y) < \min(\delta, \delta')$ we have $Z, Z' \in \mathscr{S}$ such that

$$P\left(Z, p\alpha(Y)\right) < \delta \quad , \qquad P\left(Z', p\beta(Y)\right) < \delta'$$

and one of the following holds:

(1) $\alpha(X) = \alpha(Z)$, $\beta(Z) = \alpha(Y)$, $\beta(Y) = \alpha(Z')$, $\beta(Z') = \beta(X)$;

(2) $\alpha(X) = \alpha(Z)$, $\beta(X) = \alpha(Z')$, $\beta(Z) = \alpha(Y)$, $\beta(Z') = \beta(Y)$;

(3) $\alpha(Z) = \alpha(Y)$, $\beta(Z) = \alpha(X)$, $\beta(Y) = \alpha(Z')$, $\beta(X) = \beta(Z')$;

(4) $\alpha(Z) = \alpha(Y)$, $\beta(Z) = \alpha(X)$, $\beta(X) = \alpha(Z')$, $\beta(Z') = \beta(Y)$.

In case (1) we have, by (10-5),

$$\overline{F}(X) = \overline{F}(Z) \cup \overline{F}(Y) \cup \overline{F}(Z') \quad .$$

Then $\overline{F}(X) \supset \overline{F}(Y)$. Since

$$\overline{F}(Z) \subset \overline{F}\, p\, \alpha(Y) \oplus \left[-\varepsilon/2,\ \varepsilon/2\right] \subset \overline{F}(Y) \oplus \left[-\varepsilon/2,\ \varepsilon/2\right] \quad ,$$

and similarly,

$$\overline{F}(Z') \subset \overline{F}(Y) \oplus \left[-\varepsilon/2,\ \varepsilon/2\right] \quad ,$$

it follows that

$$\overline{F}(X) \subset \overline{F}(Y) \oplus \left[-\varepsilon, \varepsilon\right] .$$

Hence $P\left(\overline{F}(X), \overline{F}(Y)\right) < \varepsilon$.

In case (2), we have, by (10-5),

$$\overline{F}(X) \cup \overline{F}(Z') = \overline{F}(Z) \cup \overline{F}(Y) \quad .$$

Since $\overline{F}(Z) \subset \overline{F}(Y) \oplus \left[-\varepsilon/2, \varepsilon/2\right]$ and $\overline{F}(Z') \subset \overline{F}(Y) \oplus \left[-\varepsilon/2,\ \varepsilon/2\right]$, it follows that $\overline{F}(X) \subset \overline{F}(Y) \oplus \left[-\varepsilon, \varepsilon\right]$. On the other hand,

$$P\left(\overline{F}(Z'), \overline{F}\, p\, \beta(X)\right) \leqslant P\left(\overline{F}(Z'), \overline{F}\, p\, \beta(Y)\right) \oplus P\left(\overline{F}\, p\, \beta(X), \overline{F}\, p\, \beta(Y)\right)$$

$$< \varepsilon/4 + \varepsilon/4 = \varepsilon/2$$

so that

$$\overline{F}(Z') \subset \overline{F}(X) \oplus \left[-\varepsilon/2,\ \varepsilon/2\right] \quad .$$

Hence $\overline{F}(Y) \subset \overline{F}(X) + \left[-\varepsilon, \varepsilon\right]$. This again proves that $P\left(\overline{F}(X), \overline{F}(Y)\right) < \varepsilon$.

Similar argument shows that our assertion also holds for the other two cases. q.e.d.

(10-7) Let I be an element of $\mathcal{J}$ and let $f : I \rightarrow R$ be a continuous function. Let $\{F_n\}$ be a sequence of arithmetic functions with $\lim_{n\rightarrow\infty} F_n = f$. Then

$$\overline{F}_1(I) \supset \overline{F}_2(I) \supset \ldots$$

and $\bigcap_{n=1}^{\infty} \overline{F}_n(I) = f(I)$.

Proof. Let n be an integer $\geqslant 1$. Let $\varepsilon > 0$. For every $x \in \mathcal{J}$ there is an $r_x > 0$ such that whenever $X \in \mathcal{J}$ with $P\bigl(X, p(x)\bigr) < r_x$, $F_n(X)$ and $F_{n+1}(X)$ are defined and $P\bigl(F_{n+1}(X), F_{n+1}p(x)\bigr) < \varepsilon$. Let δ be a positive number such that whenever $X \in \mathcal{J}_I$ with $\sigma(X) < \delta$, we have $P\bigl(X, p(x)\bigr) < r_x$ for some $x \in X$.

Let $\xi = \{\xi_1, \ldots, \xi_r\}$ be any subdivision of I with $\sigma(\xi) < \delta$. For every $i = 1, \ldots, r$, there is an $x_i \in I$ such that $P\bigl(\xi_i, p(x_i)\bigr) < r_{x_i}$. Then

$$F_{n+1}(\xi_i) \subset F_{n+1}\, p(x_i) \oplus [-\varepsilon, \varepsilon] \subset F_n\, p(x_i) \oplus [-\varepsilon, \varepsilon]$$

$$F_n(I, \xi) \oplus [-\varepsilon, \varepsilon]$$

so that

$$F_{n+1}(I, \xi) \subset F_n(I, \xi) \oplus [-\varepsilon, \varepsilon] .$$

Hence

$$\overline{F}_{n+1}(I) \subset \overline{F}_n(I) \oplus [-\varepsilon, \varepsilon] .$$

Since ε is arbitrary, it follows that

$$\overline{F}_{n+1}(I) \subset \overline{F}_n(I) \quad .$$

This proves that

$$\overline{F}_1(I) \supset \overline{F}_2(I) \supset \ldots \quad .$$

Clearly $\overline{F}_n(I) \supset f(I)$ for all n . Let $\varepsilon > 0$.

For every $n = 1, 2, \ldots$ we let

$$I_n = \left\{ x \in I \;\middle|\; P\left(F_n p(x), pf(x)\right) \geqslant \varepsilon \right\} .$$

It follows from the continuity of F_n that I_n is closed in I . Since $I_1 \supset I_2 \supset \ldots$ and $\bigcap_{n=1}^{\infty} I_n = \emptyset$, we infer that there is an integer n_o such that whenever $n > n_o$, $P\left(F_n p(x), pf(x)\right) < \varepsilon$ holds for all $x \in I$. Let $n > n_o$. Then for any subdivision $\xi = \left\{\xi_1, \ldots, \xi_r\right\}$ of I we have

$$F_n(\xi_i) \subset f(\xi_i) \oplus \left[-\varepsilon, \varepsilon\right] \quad , \qquad i = 1, \ldots, r \quad .$$

Hence

$$\overline{F}_n(I) \subset F_n(I, \xi) \subset f(I) \oplus \left[-\varepsilon, \varepsilon\right] \quad .$$

Since ε is arbitrary, we infer that

$$\bigcap_{n=1}^{\infty} \overline{F}_n(I) = f(I) \qquad \text{q.e.d.}$$

Theorem 3. <u>Let</u> $I \in \mathcal{J}$ <u>and let</u> $f : I \rightarrow R$ <u>be a continuous function.</u> <u>Let</u> $\left\{F_n\right\}$ <u>be a sequence of arithmetic functions with</u> $\lim_{n \to \infty} F_n = f$.

Then there is a sequence $\xi^{(1)} \prec \xi^{(2)} \prec \dots$ *of subdivisions of* I *such that* $\lim_{n\to\infty} \sigma\left(\xi^{(n)}\right) = 0$ *and*

$$\bigcap_{n=1}^{\infty} F_n\left(I, \xi^{(n)}\right) = f(I) \quad .$$

If, moreover, for every $x \in I$ *and every integer* $n \geqslant 1$, $F_{n+1}\, p(x)$ *is contained in the interior of* $F_n\, p(x)$, *then there is a sequence* $\xi^{(1)} \prec \xi^{(2)} \prec \dots$ *of subdivisions of* I *such that* $\lim_{n\to\infty} \sigma\left(\xi^{(n)}\right) = 0$, $F_1\left(I, \xi^{(1)}\right) \supset F_2\left(I, \xi^{(2)}\right) \supset \dots$ *and* $\bigcap_{n=1}^{\infty} F_n\left(I, \xi^{(n)}\right) = f(I)$.

Proof. By (10-7), we have $\overline{F}_1(I) \supset \overline{F}_2(I) \supset \dots$ and $\bigcap_{n=1}^{\infty} \overline{F}_n(I) = f(I)$. Let $\xi^{(1)}$ be a subdivision of I such that $\sigma\left(\xi^{(1)}\right) < 1$ and $P\left(F\left(I, \xi^{(1)}\right), \overline{F}(I)\right) < 1$. Suppose that we have subdivisions $\xi^{(1)} \prec \xi^{(2)} \prec \dots \prec \xi^{(k)}$ of I such that $\sigma\left(\xi^{(n)}\right) < 1/n$ and $P\left(F\left(I, \xi^{(n)}\right), \overline{F}(I)\right) < 1/n$, $n = 1, \dots, k$. We let $\xi^{(k+1)}$ be a refinement of $\xi^{(n)}$ with $P\left(F_{k+1}\left(I, \xi^{(k+1)}\right), \overline{F}_{k+1}(I)\right) < 1/(k+1)$. By induction, we have a sequence $\xi^{(1)} \prec \xi^{(2)} \prec \dots$ of subdivisions of I with $P\left(F_n\left(I, \xi^{(n)}\right), \overline{F}_n(I)\right) < 1/n$. Hence

$$\bigcap_{n=1}^{\infty} F_n\left(I, \xi^{(n)}\right) = f(I) \quad .$$

If for every $x \in I$ and every integer $n \geqslant 1$, $F_{n+1}\, p(x)$ is contained in the interior of $F_n\, p(x)$, as in the proof of (10-7), there is a $\delta_n > 0$ such that whenever ξ is a subdivision of I with $\sigma(\xi) < \delta_n$, $F_n(I, \xi) \supset F_{n+1}(I, \xi)$, $n = 1, 2, \dots$. Now we construct a sequence

$\xi^{(1)} \prec \xi^{(2)} \prec \ldots$ of subdivisions of I , just as above, satisfying the additional condition that

$$\sigma\left(\xi^{(n)}\right) < \delta_n , \qquad n = 1, 2, \ldots$$

Then our conclusion follows. q.e.d.

Let F be an arithmetic function with domain $\mathcal{D}(F)$ and $\mathcal{J}^F$ be the subset of $\mathcal{J}$ consisting of all the elements I of $\mathcal{J}$ with $p(I) \subset \mathcal{D}(F)$. Then, by (10-4), we may define a function

$$\Sigma_F : \mathcal{J}^F \longrightarrow \mathcal{J}$$

by

$$\Sigma_F(I) = \bigcap_{n=1}^{\infty} \Sigma\left(F, I, \xi^{(n)}\right) ,$$

where $I \in \mathcal{J}^F$ and $\xi^{(1)} \prec \xi^{(2)} \prec \ldots$ is a sequence of subdivisions of I with $\lim_{n \to \infty} \sigma\left(\xi^{(n)}\right) = 0$.

As (10-5), (10-6) and (10-7), we have

(10-8) Let I and J be elements of $\mathcal{J}^F$ with $\beta(I) = \alpha(J)$. Then $I \cup J \in \mathcal{J}^F$ and

$$\Sigma_F(I \cup J) = \Sigma_F(I) \oplus \Sigma_F(J) .$$

(10-9) $\Sigma_F : \mathcal{J}^F \rightarrow \mathcal{J}$ is continuous.

(10-10) Let $I = [a,b] \in \mathcal{J}$ and let $f : I \rightarrow R$ be a continuous function. Let $\{F_n\}$ be a sequence of arithmetic functions with $\lim_{n \to \infty} F_n = f$.

Then

$$\Sigma_{F_1}(I) \supset \Sigma_{F_2}(I) \supset \dots$$

and $\bigcap_{n=1}^{\infty} \Sigma_{F_n}(I) = \int_a^b f(x)\,dx$.

Making use of (10-10) and the definition of Σ_F we can prove

Theorem 4. Let $I = [a,b] \in \mathcal{J}$ and let $f : I \to R$ be a continuous function. Let $\{F_n\}$ be a sequence of arithmetic functions with $\lim_{n\to\infty} F_n = f$. Then there is a sequence $\xi^{(1)} \prec \xi^{(2)} \prec \dots$ of subdivisions of I such that $\lim_{n\to\infty} \sigma\left(\xi^{(n)}\right) = 0$ and

$$\bigcap_{n=1}^{\infty} \Sigma\left(F_n, I, \xi^{(n)}\right) = \int_a^b f(x)\,dx \;.$$

If, moreover, for every $x \in I$ and every integer $n \geqslant 1$ $F_{n+1}p(x)$ is contained in the interior of $F_n p(x)$, then there is a sequence $\xi^{(1)} \prec \xi^{(2)} \prec \dots$ of subdivisions of I such that $\lim_{n\to\infty} \sigma\left(\xi^{(n)}\right) = 0$, $\Sigma\left(F_1, I, \xi^{(1)}\right) \supset \Sigma\left(F_2, I, \xi^{(2)}\right) \supset \dots$ and $\bigcap_{n=1}^{\infty} \Sigma\left(F_n, I, \xi^{(n)}\right) = \int_a^b f(x)\,dx$.