A Method of Estimating the Solution of a System of Ordinary Differential Equations by Using Interval Taylor Series

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I. Ellipsoid Method for Estimating the Accuracy of the Solution of a System of ODE

In [1, 2], Filippov proposed the following *ellipsoid method* for estimating the error of the approximate solution of a system of ordinary differential equations (ODE): In this method, at any moment of time t, we have an ellipsoid $E(t)$ that is guaranteed to contain the actual value of the solution $x(t) = (x_1(t), ..., x_n(t))$. The initial ellipsoid represents the initial uncertainty. The equations that describe how this ellipsoid changes in time are obtained from the original system of ODE, if we take into account rounding errors, and additional errors caused by the method itself.

The evolution equations can be then solved using an appropriate iterative technique. In particular, Filippov uses Adams' method.

II. Taylor Series Techniques Leads to Better Estimates for an Ellipsoid Method

Instead of Adams' method, we used Taylor series techniques to solve the evolution equations for the ellipsoid. As a result, for several reasonable systems of ODE, we got better error estimates.

III. We Can Get Even Better Error Estimates If We Use Parallelepipeds Instead of Ellipsoids

We implemented this method using a PASCAL-SC compiler [3] and a PASCAL-XSC compiler [4]. It turned out after processing our program, these compilers actually ended up with algorithm in which the parameters of the ellipsoid $E(t)$ were computed as follows:

- first, we compute a changing *parallelepiped* that contains the actual solution of the system, and then
- compute an ellipsoid that contains this parallelepiped.

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Because of that, we get a better estimate if we do not use ellipsoids at all, and instead, use a changing *parallelepiped* to estimate the errors of the approximate solution. Thus, we arrive at the following method:

IV. Mathematical Formulation of the Problem

Let the system of ordinary differential equations

$$x'(t) = f(t, x(t)) \qquad (1)$$

with the initial conditions

$$x(t_0) = x_0 \qquad (2)$$

be given, where $x(t)$ and $f(t, x)$ are functions whose values are vectors in $n-$dimensional Euclidean space.

Let us assume that for every initial value $x(t_0)$, the system (1) has a unique solution $x(t)$.

V. The Idea Behind Our Method

Let $\tilde{x}(t)$ be an approximate solution to the problem (1)–(2). Usually, it is a vector function, that is obtained by some simple interpolation procedure (piecewise constant or piecewise linear) from its values for $t_i = t_0 + ih$, $i = 1, .., n$ that are computed during the numerical solution (e.g., if we apply the Runge-Kutta method).

To describe how the parallelepiped evolves in time, i.e., how much error this method can add on the interval between two consequent points, we we will define the notion of a *local error*. Suppose that at the moment t_{i-1}, we know the value of $x(t)$ precisely, i.e, that $x(t_{i-1}) = \tilde{x}(t_{i-1})$. Since we assumed that the solution of our system (1) is uniquely determined by the initial conditions, the condition $x(t_{i-1}) = \tilde{x}(t_{i-1})$ uniquely determines a solution; we will denote this solution by $x_{i-1}(t)$. As t goes from t_{i-1} to t_i, the value $x_{i-1}(t)$ of this solution may differ from $\tilde{x}(t)$. This difference $l(t) = \tilde{x}(t_i) - x_{i-1}(t_i)$ will be called the *local error*.

The local error can be estimated as a sum of the error of the approximation method (that is usually known) and the rounding error.

Our goal is to estimate the difference between the exact solution and the approximate solution of ODE system (1):

$$e(t) = x(t) - \tilde{x}(t) \qquad (3).$$

For t from t_{i-1} to t_i, the desired difference $e(t)$ can be represented as a difference

$$e(t) = y(t) - l(t)$$

between a function

$$y(t) = x(t) - x_{i-1}(t) \qquad (4)$$

and the local error $l(t)$. Since we already know the bounds for local error, to estimate $e(t)$, it is sufficient to be able to estimate $y(t)$.

Since both $x(t)$ and $x_{i-1}(t)$ are solutions of the equation (1), after differentiating both sides of the equation (4), we get the following equation:

$$y'(t) = x'(t) - x'_{i-1}(t) = f(t, x(t)) - f(t, x_{i-1}(t)).$$

Substituting $x(t) = x_{i-1}(t) + y(t)$ into this equation, we obtain the first order differential equation for y:

$$y'(t) = f(t, x_{i-1}(t) + y(t)) - f(t, x_{i-1}(t)). \qquad (5)$$

The initial value for y is $y(t_{i-1}) = x(t_{i-1}) - x_{i-1}(t_{i-1})$. By definition of x_{i-1}, this difference is equal to $y(t_{i-1}) = x(t_{i-1}) - \tilde{x}(t_{i-1})$. So, for $i = 0$, we have $y(t_0) = 0$, and for further i, as an estimate for this initial value, we can take the error bound that was obtained on the previous step of the error estimation.

To solve the system (5), we use the method of Taylor series [5] (we use Lohner's method to take the remainder into account).

VI. Testing

The method of parallelepipeds was successfully tested on two simple ODE systems:

$$x'_1(t) = x_1 - x_2,\ x'_2(t) = 2x_1 - x_2^3,$$

$$x_1(0) = 0.25,\ x_2(0) = 0,$$

and

$$x'_1(t) = -x_2 + x_1 \cdot (x_1^2 + x_2^2 + 1),$$

$$x'_2(t) = x_1 + x_2 \cdot (x_1^2 + x_2^2 + 1),$$

$$x_1(0) = 0.5,\ x_2(0) = 0.$$

VII. Potential Applications

We are currently applying our method to estimating the phase state of dynamical systems [7], in particular, to intertial navigation problems.

References

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