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A Posteriori Estimation of the Solution of a System of Ordinary Differential Equations with the Help of Taylor Series

V. S. Zyuzin and O. V. Mushtakova

I. Description of a Problem

Suppose that we have a system of ordinary differential equations, and we know the initial conditions. The majority of the existing numerical methods for solving this system (e.g., Ringe-Kutta method) provides us only with an *approximate* solution, but these method give no guaranteed estimate of the corresponding error.

In [1, 2], a class of methods (called *a posteriori* methods) was proposed for estimating errors of approximate solutions of systems of ordinary differential equations (ODE) with known initial conditions. These methods consist of the following three steps:

1 An approximate solution must be found by some standard method.

> The authors of [1, 2] recommend to use Runge-Kutta method, or other techniques based on replacing a differential equation with an equation in finite differences.

2 Then, we compute the *residual* for this approximate solution, i.e., the difference between the left- and right-hand sides of the equation.

3 Finally, we use this residual to find two-sided boundaries for the solution of the system of ODE.

These methods give a guaranted estimate for the solution's error; in other words, they give us an interval that is guaranteed to contain the (unknown) exact solution.

The main problem with this method is that it often "overshoots": the resulting intervals are too wide.

II. Main Idea

Our main idea is to use a *different* method for computing the approximate solution:

- instead of the Runge-Kutta method (or other finite difference methods),
- we propose to use methods based on *Taylor series.*

The authors are with the Department of Mechanics and Mathematics, Saratov State University, Astrakhanskaya 83, Saratov, 410071, Russia, e-mail zyuzin@scnit.saratov.su.

The idea of the Taylor-series based methods is well known:

- we express the (unknown) solution as a Taylor series with unknown coefficients;
- we substitute these expressions into the differential equations; as a result, we get a system of polynomial equalities;
- polynomials are equal iff all their cofficients are equal; so, we arrive at an (algebraic) system of equations, in which the unknowns are the coefficients of the desired Taylor expansions;
- finally, we solve this system, and find the values of the coefficients; substituting the resulting values of the coefficients into the expressions for the Taylor series, we get the desired approximate solution.

This modification turns out to be very efficient, especially in cases when in addition to estimating the error of the approximate solution, we are also interested in the value of the residual.

Comment. Although Taylor series have been used for solving differential equations, they have never been used as a part of an a posteriori method.

III. Test Problems

We have used several test problems (for which exact solutions are known) to compare our method with the method from [1, 2]. First, we applied both methods to the following two *linear* systems:

1) $y' = y + 2$, $y_1' = y_1 + 1$, $y_2' = y_2$,
$y(0) = 1$, $y_1(0) = 0$, $y_2(0) = 1$.
For this system, the exact solution is known:
$y = 3 \cdot e^t - 2$, $y_1 = e^t - 1$, $y_2 = e^t$.

2) $y_1' = y_2$, $y_1(0) = 0$, $y_2' = -y_1$, $y_2(0) = 1$,
$u_1' = u_2 + 1$, $u_1(0) = 0$, $u_2' = u_1 + 1$, $u_2(0) = 0$,
$v_1' = v_2$, $v_1(0) = 0$, $v_2' = v_1$, $v_2(0) = 0$.
For this system, the exact solution is:
$y_1(t) = \sin(t)$, $y_2(t) = \cos(t)$,
$u_1(t) = e^t - 1$, $u_2(t) = e^t - 1$,
$v_1(t) = C_1 \cdot e^{-t} + C_2 \cdot e^t$, $v_2(t) = -C_1 \cdot e^{-t} + C_2 \cdot e^t$,
where
$C_1 = e^h/2 \cdot (v_1(h) - v_2(h))$ and

$$C_2 = e^{-h}/2 \cdot (v_1(h) + v_2(h)).$$

Both algorithms were implemented in PASCAL-XSC [3]. For both examples, the error estimates that we obtained using Taylor series method were strictly contained in the two-sided bounds obtained by using a method from [1, 2] (where the approximate solution is obtained by using Runge-Kutta techniques).

As a third test problem, we considered the following *non-linear* system:

3) $y' = y^2$, $y_1' = y_1^2 + 1$, $y_2' = y_2^2$,
$y(0) = -1$, $y_1(0) = 0$, $y_2(0) = 0$.
For this system, the exact solution is:
$y(t) = -1/(t+1)$, $y_1(t) = tg(t)$, $y_2(t) = 0$.

Comment. In the non-linear case, in contrast to the linear one, it is necessary to take into account additional functions and a priori given constants from [1, 2].

IV. Applications

We have applied this method to airplane *inertial navigation* [4]. As a result of this applications, we obtained two-sided polynomial approximations that contain the (unknown) exact solution of the corresponding system of differential equations. The resulting error is $\leq 10^{-8}$.

We are planning to use this method for several other applications.

References

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