In Expert Systems, Even If We Fix AND/OR Operations, A Natural Answer To A Composite Query Is The Interval Of Possible Degrees of Belief

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***Abstract*— In expert systems, even if we fix AND and OR operations, the degree of belief $d(Q)$ in a composite query Q can take different values depending on how exactly we represent the query in terms of AND, OR, and NOT. In this paper, we compute the interval of possible values of $d(Q)$.**

I. Introduction

A. Uncertainty Must Be Represented In Expert Systems

The goal of an expert system is to answer queries. To be able to do that, the expert system must contain the experts' knowledge about the domain of interest (be it medicine, physics, etc). Experts can usually formulate this knowledge as a set of statements.

Experts are often uncertain about their statements. The degree of expert's belief in his statement A is usually described by a number $d(A)$ from the interval $[0,1]$ (1 corresponds to complete belief, 0 to complete disbelief).

B. Degree Of Belief In A Query: The Simplest Case

The simplest case for an expert system is when the query Q coincides with one of the statements that is already in the knowledge base. Then, as an answer to that query, we can return the degree of belief $d(Q)$ that is stored in this expert system.

C. Degree of Belief In Composite Queries: A Problem

A more complicated case is when we want to find the degree of belief $d(Q)$ in a query Q that is a logical combination of the statements from the knowledge base (or for a query Q that can be shown to be equivalent to such a combination).

Even in the simplest of these cases, when, e.g., a query Q is a formula of the type $A\&B$ with A and B both in the knowledge base, we already have a problem: Indeed, if we know the degrees of belief (subjective probabilities) $a = d(A)$ and $b = t(B)$ of the statements A and B, then the actual degree of belief in $A\&B$ can be $d(A) \cdot d(B)$ (if A and B are random and uncorrelated), or $\min(d(A), d(B))$ (if one of the statements A and B is a corollary of the other one), etc. The only thing that we know for sure about the degree of belief of $A\&B$ is that it lies between $\min(a+b-1, 0)$ and $\min(a, b)$. Therefore, even if we start with precise values, e.g., $a = 0.3$ and $b = 0.4$, we get an *interval* $[0, 0.3]$ of possible values of $d(A\&B)$.

D. AND/OR operations

To avoid the above-described ambiguity, usually, in expert systems, a *single* value from the interval of the possible values of $d(A\&B)$ is chosen as an answer to the query "$A\&B$?". This chosen value depends only on the values of $d(A)$ and $d(B)$, so, from the mathematical viewpoint, it is a *function* of $d(A)$ and $d(B)$. This function $f_\&$ is called an *AND operation.*

Similarly, in expert systems, two other functions are defined:

- A function $f_\vee(a,b)$ called an *OR operation* that transforms the values of $d(A)$ and $d(B)$ into a single value $f_\vee(d(A), d(B))$ that is produced as $d(A \vee B)$.
- A function $f_\neg(a)$ called a *NOT operation* that transforms the value of $d(A)$ into a degree of belief $f_\neg(d(A)) = d(\neg A)$ in "not A".

The simplest possible AND, OR, and NOT operations are min, max, and $f_\neg(a) = 1 - a$ (for which $d(A) \rightarrow d(\neg A) = 1 - d(A)$).

E. Even If AND and OR Operations Are Fixed, For More Complicated Queries, We Still Have Several Possible Degrees of Belief

If we fix AND, OR, and NOT operations, then we can in principle, knowing the degree of belief in the

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basic statements, determine the degree of belief in their logical combination Q. To do that, we represent the given formula Q as a combination of &, $\vee$, and $\neg$, and then consequently use our chosen operations with degrees of belief instead of these logical symbols.

For example, a formula Q of the type $A \rightarrow B$ can be represented as $B \vee \neg A$, and therefore, as a degree of belief in Q, we can take $f_\vee(d(B), f_\neg(d(A))$.

The problem with this idea is that for a complicated formula Q, e.g., $A \rightarrow B$, there exists several methods of representing $A \rightarrow B$ in terms of &, $\vee$, and $\neg$: $B \vee \neg A$, $(A\&B) \vee \neg A$ etc. These formulas are equivalent in classical logic, but they lead to different numerical values for $d(Q)$.

F. What We Are Going To Do: Main Result

In this paper, we describe the interval of possible values of $d(Q)$ for a given formula Q. Namely, we give a description for the case when $f_\& = \min$ and $f_\vee = \max$, and we show that for other choices of AND and OR operations, for practically any formula Q, the corresponding interval simply coincides with $[0, 1]$ (and therefore, makes no sense).

G. What We Are Going To Do: Additional Result

The assumption that we can describe the initial degrees of belief by numbers is an idealization. In reality, an expert cannot describe his degree of belief beyond a certain accuracy. No one is able to say: "I believe in this with probability 83.4%". Therefore, a more realistic description of the initial degrees of belief is by *intervals*.

For this case, similar bounds on $d(Q)$ are obtained.

H. Methods

The methods that we use are a generalization if the methods used in [2] for formulas with two variables.

II. Definitions and The Main Results

Let us start with the basic definitions of propositional logic.

Definition 1. Assume that a finite set

$$V = \{v_1, ..., v_n\}$$

is given. Its elements v_i will be called *fuzzy variables.* By a *literal*, we will mean either a variable v_i or its negation $\neg v_i$. A *propositional formula* with n variables v_i is then defined as follows:

- every variable v_i is a formula;
- if A is a formula, then $\neg A$ (called *"not A"*) is also a formula;
- if A and B are formulas, then the following are formulas:
  - $A\&B$ (called *"A and B"*);
  - $A \vee B$ (called *"A or B"*);
- nothing else is a formula.

Formulas that have been used in the construction of a formula A are called its *subformulas.* Formulas different from variables are called *composite* formulas.

Definition 2. By a *Boolean vector*, we mean a mapping $t : V \rightarrow \{T, F\}$, where T and F are special symbols that stand for "true" and "false". In the following text, we will identify T with 1, and F with 0. If t is given, then we can define $t(A)$ for an arbitrary composite propositional formula A as follows:

- $t(\neg A) = \neg(t(A))$;
- $t(A\&B) = t(A)\&t(B)$;
- $t(A \vee B) = t(A) \vee t(B)$.

We say that formulas A and B are *equivalent* in propositional logic if $t(A) = t(B)$ for every Boolean vector t.

Comment. For degrees of belief, we can have similar definitions.

Definition 3. By the *initial degrees of belief*, we mean a function $d : V \rightarrow [0, 1]$.

Definition 4. Assume that we are given three functions:

- $f_\& : [0, 1] \times [0, 1] \rightarrow [0, 1]$,
- $f_\vee : [0, 1] \times [0, 1] \rightarrow [0, 1]$, and
- $f_\neg : [0, 1] \rightarrow [0, 1]$.

Suppose that the initial degrees of belief d are given. Then, we can define $d(A)$ for an arbitrary composite propositional formula A as follows:

- $d(\neg A) = f_\neg(d(A))$;
- $d(A\&B) = f_\&(d(A), d(B))$;
- $d(A \vee B) = f_\vee(d(A), d(B))$.

Definition 5. Let A be a propositional formula, and let d be the initial degrees of belief. By S, we denote the set of values of $d(B)$ for all formulas B that are equivalent to A in propositional logic. Let us denote $\sup S$ by $d^+(A)$ and $\inf S$ by $d^-(A)$.

Comment. Our goal is to describe the interval $[d^-(A), d^+(A)]$. In order to describe this interval for min and max, we will need the following definition:

Definition 6. Let A be a propositional formula.

- We say that a Boolean vector *satisfies* the formula A if $t(A) = T$.
- For every Boolean vector t, we define a formula $C(t)$ as $a_1\&...\&a_n$, where:
  - $a_i = v_i$ if $t(v_i) = T$, and

- $a_i = \neg v_i$ if $t(v_i) = F$.

- By a *complete disjunctive normal form* of the formula A (or, cDNF(A), for short), we mean the formula $C(t_1) \vee C(t_2) \vee ...$, where $t_1, t_2, ...$ are all Boolean vectors that satisfy the formula A.

Comment. The formula cDNF(A) means that the variables $v_1, ..., v_n$ take one of the tuples of values that make the formula A true. Therefore, this formula cDNF(A) is equivalent to A in propositional logic.

Definition 7. Let A be a propositional formula.

- We say that a Boolean vector *does not satisfy* the formula A if $t(A) = F$.
- For every Boolean vector t, we define a formula $D(t)$ as $a_1 \vee ... \vee a_n$, where:
  - $a_i = \neg v_i$ if $t(v_i) = T$, and
  - $a_i = v_i$ if $t(v_i) = F$.
- By a *complete conjunctive normal form* of the formula A (or, cCNF(A), for short), we mean the formula $D(t_1)\&D(t_2)\&...$, where $t_1, t_2, ...$ are all Boolean vectors that satisfy the formula A.

Comment. The formula cCNF(A) means that the variables $v_1, ..., v_n$ do not take one of the tuples of values that make the formula A false. This formula is therefore equivalent to A in propositional logic.

THEOREM 1. *Assume that $f_\& = \min$, $f_\vee = \max$, and $f_\neg(a) = 1 - a$. Then, for an arbitrary propositional formula A, $d^-(A) = d(\text{cDNF}(A))$ and $d^+(A) = d(\text{cCNF}(A))$.*

Comments.

1. All the proofs are given in the last section.
2. For operations different from & and $\vee$, the interval $[d^-(A), d^+(A)]$ becomes trivial. To formulate the corresponding result in precise terms, let is recall the following definitions and results [5, 3]:

Definition 8. [3] By a *continuous, strictly increasing t-norm*, we mean a function $\& : [0,1] \times [0,1] \to [0,1]$ that satisfies the following conditions:

(1) *Boundary conditions:* $x\&0 = 0\&x = 0$, $x\&1 = 1\&x = x$.
(2) *Commutativity:* $x\&y = y\&x$.
(3) *Associativity:* $x\&(y\&z) = (x\&y)\&z$.
(4) *Monotonicity:* if $a_1 \le a_2$ and $b_1 \le b_2$, then $a_1\&b_1 \le a_2\&b_2$.
(5) *Strict monotonicity:* if $a_1 < a_2$ and $b_1 < b_2$, then $a_1\&b_1 < a_2\&b_2$.
(6) *Continuity:* the function & is continuous in both variables.

Definition 9. [3] By a *continuous, strictly increasing t-conorm*, we mean a function

$$\vee : [0,1] \times [0,1] \to [0,1]$$

that satisfies the following conditions:

(1) *Boundary conditions:* $x \vee 0 = 0 \vee x = x$, $x \vee 1 = 1 \vee x = 1$.
(2) *Commutativity:* $x \vee y = y \vee x$.
(3) *Associativity:* $x \vee (y \vee z) = (x \vee y) \vee z$.
(4) *Monotonicity:* if $a_1 \le a_2$ and $b_1 \le b_2$, then $a_1 \vee b_1 \le a_2 \vee b_2$.
(5) *Strict monotonicity:* if $a_1 < a_2$ and $b_1 < b_2$, then $a_1 \vee b_1 < a_2 \vee b_2$.
(6) *Continuity:* the function $\vee$ is continuous in both variables.

CLASSIFICATION THEOREM FOR AND OPERATIONS. [3] *Any continuous, strictly monotonic t-norm can be represented as $a\&b = \varphi^{-1}(\varphi(a) \cdot \varphi(b))$ for some continuous and strictly increasing function $\varphi : [0,1] \to [0,1]$ for which $\varphi(0) = 0$ and $\varphi(1) = 1$.*

CLASSIFICATION THEOREM FOR OR OPERATIONS. [3] *Any continuous, strictly monotonic t-conorm $\vee$ can be represented as $a \vee b = \psi^{-1}(\psi(a) \cdot \psi(b))$ for some continuous and strictly decreasing function $\psi : [0,1] \to [0,1]$ for which $\psi(0) = 1$ and $\psi(1) = 0$.*

Definition 10. Let d be the initial degrees of belief. We say that a propositional formula A is *not completely certain* if $d(A) \in (0,1)$ (i.e., if $d(A) \ne 0$ and $d(A) \ne 1$).

Comment. In the following text, we will assume that a negation operation $f_\neg$ is a continuous strictly decreasing function $[0,1] \to [0,1]$ that maps 0 into 1 and 1 into 0.

THEOREM 2. *Let $f_\&$ is a continuous, strictly monotonic t-norm, and let d be the initial degrees of belief for which a formula A is not completely certain. Then $[d^-(A), d^+(A)] = [0,1]$.*

THEOREM 3. *Let $f_\vee$ is a continuous, strictly monotonic t-conorm, and let d be the initial degrees of belief for which a formula A is not completely certain. Then $[d^-(A), d^+(A)] = [0,1]$.*

III. Additional Result: Case When Initial Degrees of Belief Are Intervals

In view of Theorems 2 and 3, in this Section, we will only consider the case when $f_\& = \min$, $f_\vee = \max$, and $f_\neg(a) = 1 - a$. These operations can be naturally extended to intervals in the following way:

Definition 11. Let $f : [0,1] \times [0,1] \to [0,1]$ be a continuous function, and let I denote the set of all subintervals of the interval $[0,1]$. Then, for $\mathbf{a} \in I$ and $\mathbf{b} \in I$, we define $f(\mathbf{a},\mathbf{b})$ as the the interval $\{f(a,b) \,|\, a \in \mathbf{a}, b \in \mathbf{b}\}$.

Comment. For the above operations, this extension is easy to describe:

PROPOSITION. *For all $a^- \leq a^+$ and $b^- \leq b^+$, the following formulas are true:*

$$\min([a^-,a^+],[b^-,b^+]) = [\min(a^-,b^-),\min(b^-,b^+)];$$

$$\max([a^-,a^+],[b^-,b^+]) = [\max(a^-,b^-),\max(b^-,b^+)];$$

$$1-[a^-,a^+] = [1-a^+,1-a^-].$$

Definition 12. By the *initial interval degrees of belief*, we mean a function $\mathbf{d}: V \to I$.

Definition 13. Suppose that the initial interval degrees of belief $\mathbf{d}$ are given. Then, we can define $\mathbf{d}(A)$ for an arbitrary composite propositional formula A as follows:

- $\mathbf{d}(\neg A) = 1 - \mathbf{d}(A)$;
- $\mathbf{d}(A \& B) = \min(\mathbf{d}(A),\mathbf{d}(B))$;
- $\mathbf{d}(A \vee B) = \max(\mathbf{d}(A),\mathbf{d}(B))$.

Definition 14.

- We say that a degree of belief $[a^-,a^+]$ is *smaller* that the degree of belief $[b^-,b^+]$ (and denote it by $[a^-,a^+] \leq [b^-,b^+]$) if $a^- \leq b^-$ and $a^+ \leq b^+$.
- We say that a formula A is *weaker* than a formula B (and that B is *stronger* than A) if for every initial interval degrees of belief $\mathbf{d}$, we have $\mathbf{d}(A) \leq \mathbf{d}(B)$.

THEOREM 4. *Every propositional formula A is stronger than cDNF(A) and weaker than cCNF(A).*

Comment. If formulas A and B are equivalent in propositional logic, then cCNF(A)=cCNF(B) and cDNF(A)=cDNF(B). So, if A and B are equivalent, then the degree of belief in B is always bounded by the degrees of belief in cDNF(A) and in cCNF(A). This theorem is therefore, a direct interval analogue of Theorem 1.

IV. Proofs

A. Proof of Theorem 1

A.1. Main Idea

The main idea of our proof is as follows: to every formula A, we will apply some transformations that leave $d(A)$ unchanged. As a result of these transformations, we will be able to reduce A to a special form [1], and in that form, comparison with cCNF or cDNF will be easy.

We will only describe the proof for cDNF. The proof for cCNF is similar.

A.2. Moving Negations Inside

For min and max, the following Lemma can be easily proven:

LEMMA 1. *For arbitrary propositional formulas A and B,*

- $d(\neg(A \& B)) = d((\neg A) \vee (\neg B))$;
- $d(\neg(A \vee B)) = d((\neg A) \& (\neg B))$;
- $d(\neg(\neg A)) = d(A)$.

Comment. Indeed, e.g., the first statement easily follows from the statement that for all a and b, $1-\min(a,b) = \min(1-a,1-b)$, if we take $a = d(A)$ and $b = d(B)$. This statement, in its turn, can be easily proven by considering two possible cases: $a \leq b$ and $a > b$.

Using this Lemma, we can move all negations inside the formula until we have a formula with the same values of d in which negations are only applied to variables.

A.3. Moving ANDs Inside

LEMMA 2. *For arbitrary propositional formulas A, B, and C, $d(A \& (B \vee C)) = d((A \& B) \vee (A \& C))$.*

The proof of this Lemma can be reduced to a statement that for all a, b, and c, we have

$$\min(a,\max(b,c)) = \max(\min(a,b),\min(a,c)).$$

This statement can be proven by considering all possible orderings of a, b, and c.

Due to this lemma, whenever we have AND outside a composite formula, we can move this AND inside. We can repeat this operation every time when there is an AND outside the composite formula, thus moving AND further and further inside. As a result, we can get a formula B that is equivalent to A, and in which AND is applied only to the variables or to their negations.

In other words, we get a formula B of the type $C_1 \vee \ldots \vee C_k$, where each of the subformulas C_j is of the type $a \& \ldots \& b$, and $a,\ldots,b$ are *literals* (i.e., variables or their negations). Such a form is called a *Disjunctive Normal Form* (or, *DNF*, for short). Subformulas C_j are called *conjunctions*.

Due to Lemmas 1 and 2, we have $d(A) = d(B)$ for all initial degrees of belief d.

A.4. Final Proof

Let us now prove the Theorem. Since $d(A) = d(B)$, in order to prove that $d(A) \geq d(\text{cDNF}(A))$, it is now sufficient to prove that $d(B) \geq d(\text{cDNF}(A))$. Here, $\text{cDNF}(A) = C(t_1) \vee C(t_2) \vee \ldots$, therefore, $d(\text{cDNF}(A)) = \max(d(C(t_1)), d(C(t_2)), \ldots)$. So, to prove that $d(B) \geq d(\text{cDNF}(A)) =$

$\max(d(C(t_1)), d(C(t_2)), ...)$, it is sufficient to prove that $d(C(t_j)) \le d(B)$ for all j.

The proof that $d(C(t_j)) \le d(B)$ can be obtained via the following sequence of logical steps:

- By definition of cDNF, each t_j is a satisfying vector for a formula A (i.e., $t_j(A) = T$).
- Transformations from Lemmas 1 and 2 are equivalent in propositional logic. So, B is equivalent to A in propositional logic, and hence, $t_j(B) = t_j(A) = T$. So, for the Boolean vector t_j, the formula $B = C_1 \vee ... \vee C_k$ is true.
- By definition of OR, the fact that $C_1 \vee ... \vee C_k$ is true means that one of the subformulas $C_l = a\&...\&b$ is true.
- By definition of AND, the fact that $a\&...\&b$ is true means that all the literals $a, ..., b$ from the subformula C_l are true in t_j.
- By definition of $C(t_j)$, if a literal is true in t_j, then it is an element of $C(t_j)$. So, all literals $a, ..., b$ from C_l are elements of $C(t_j)$. Hence, $C(t_j)$ contains all literals from C_l (and, maybe, some other literals), i.e., either $C(t_j) = C_l$, or $C(t_j) = C_l\&c\&...$.
- In the first case,

$$d(C(t_j)) = d(C_l) \le \max(..., d(C_l), ...) = d(B).$$

- In the second case,

$$d(C(t_j)) = \min(d(C_l), d(c), ...) \le d(C_l) \le$$

$$\max(..., d(C_l), ...) \le d(B).$$

- In both cases, $d(C(t_j)) \le d(B)$.

Hence,

$$d(\mathrm{cDNF}(A)) = \max(d(C(t_1)), d(C(t_2)), ...) \le$$

$$d(B) = d(A),$$

and $d(\mathrm{cDNF}(A)) \le d(A)$. Q.E.D.

B. Proof of Theorem 2

Since A is not completely certain, we have $d(A) \ne 0$ and $d(A) \ne 1$. For every k, the formula $A\&...\&A$ (k times) is equivalent to A in propositional logic. According to the Classification Theorem for AND operations, the degree of belief in this formula is equal to $\varphi^{-1}((\varphi(a))^k)$ for the monotonic continuous function φ that corresponds to the chosen t-norm. When $k \to \infty$, we have $(\varphi(a))^k \to 0$, and therefore, $d(A\&...\&A) = \varphi^{-1}((\varphi(a))^k) \to 0$. Therefore, the set S includes values that are arbitrarily close to 0, and its infimum $d^-(A)$ is 0.

Let us now prove that it supremum is 1. Indeed, for each k, the formula $\neg((\neg A)\&...\&(\neg A))$ (k times) is also equivalent to A. Since $d(A) \in (0,1)$, and $f_\neg$ is strictly decreasing, we have $d(\neg A) \in (0,1)$ and therefore, $d((\neg A)\&...\&(\neg A)) \to 0$. Hence, $d(\neg((\neg A)\&...\&(\neg A))) \to 1$, and $d^+(A) = 1$. Q.E.D.

C. Proof of Theorem 3

This proof is similar to the proof of Theorem 2.

D. Proof of Theorem 4

This proof is similar to the proof of Theorem 1, with the only difference that instead of the relation $\le$ between numbers, we will have to use the above-defined relation $\le$ between intervals, and instead Lemmas 1 and 2, we will have to use their interval analogues [1]:

LEMMA 3. *For arbitrary propositional formulas A and B,*

- $\mathbf{d}(\neg(A\&B)) = \mathbf{d}((\neg A) \vee (\neg B))$;
- $\mathbf{d}(\neg(A \vee B)) = \mathbf{d}((\neg A)\&(\neg B))$;
- $d(\neg(\neg A)) = d(A)$.

LEMMA 4. *For arbitrary propositional formulas A, B, and C,* $\mathbf{d}(A\&(B \vee C)) = \mathbf{d}((A\&B) \vee (A\&C))$.

These Lemmas can also be proven by considering all possible cases of relationship between $a^\pm$, $b^\pm$, and $c^\pm$, where:

- by $a^\pm$, we denoted the endpoints of the interval $\mathbf{d}(A) = [a^-, a^+]$;
- similarly, $\mathbf{d}(B) = [b^-, b^+]$; and
- $\mathbf{d}(C) = [c^-, c^+]$.

Q.E.D.

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