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Description of Strictly Monotonic Interval AND/OR Operations

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***Abstract*— In expert systems, AND and OR operations are usually defined for degrees of belief that are real numbers. Such AND/OR operations have been classified by B. Schweizer and A. Sklar [1] and S. Weber [2].**

In practice, however, it is difficult to assign an exact numerical value to the expert's degree of belief. At best, we can get an *interval* of possible values. It is therefore necessary to define AND/OR operations for intervals.

In this paper, we classify such operations.

I. Degrees of Belief

The goal of an expert system is to the experts' way of making decisions. For that purpose, an expert system includes a *knowledge base* that contains the knowledge of the experts. Some statements from the knowledge base are believed to be absolutely correct. However, in their decisions, experts also use other statements, about which they are not 100% sure that they are correct. To describe the expert knowledge adequately, we must therefore store in the knowledge base not only the statements themselves, but also the indication of the extent to which the experts believe in these statements.

This *degree of belief* is usually described by a number from the interval [0,1]. An expert's degree of belief $d(A)$ in a statement A can be determined, if, e.g., we ask an expert to estimate his degree of belief on a scale from 0 to 10. If he selects 8, then we take $d(A) = 8/10$.

II. AND and OR Operations

Suppose now that we know the degrees of belief $d(A)$ and $d(B)$ in statements A and B, and we know nothing else about A and B. Suppose also that we are interested in the degree of belief of the composite statement $A\&B$. Since the only information available consists of the values $d(A)$ and $d(B)$, we must compute $d(A\&B)$ based on these values. We must be able to do that for arbitrary values $d(A)$ and $d(B)$. Therefore, we need a *function* that transforms the values $d(A)$ and $d(B)$ into an estimate for $d(A\&B)$. Such a function is called an *AND-operation.* If an AND-operation

$$T : [0,1] \times [0,1] \to [0,1]$$

is fixed, then we take $T(d(A), d(B))$ as an estimate for $d(A\&B)$.

Similarly, to estimate the degree of belief in $A \vee B$, we need an *OR-operation* $S : [0,1] \times [0,1] \to [0,1]$.

Histortically the first AND and OR operations were introduced by L. Zadeh in [3]: $T(x, y) = \min(x, y)$ and $S(x, y) = \max(x, y)$. Since then, several other operations have been proposed:

- "Soft AND" by Bellman and Zadeh [4];
- "Bold AND" and "Bold OR" by Giles [5];
- "Compensatory AND" by Zimmerman and Zesno [6];
- Yager [7] studied a general class of fuzzy connectives.

A complete classification of all AND and OR operations that satisfy several natural conditions was given by Schweizer and Sklar [1], and by Asina, Trillas and Valverde [8]. Since we will use their results, we will briefly recall them here (in slightly different notations).

III. Classification Of AND And OR Operations

Definition 1. By a *continuous, strictly increasing t-norm*, we mean a function $\& : [0,1] \times [0,1] \to [0,1]$ that satisfies the following conditions:

(1) *Boundary conditions:* $x\&0 = 0\&x = 0$, $x\&1 = 1\&x = x$.
(2) *Commutativity:* $x\&y = y\&x$.
(3) *Associativity:* $x\&(y\&z) = (x\&y)\&z$.
(4) *Monotonicity:* if $a_1 \le a_2$ and $b_1 \le b_2$, then $a_1\&b_1 \le a_2\&b_2$.
(5) *Strict monotonicity:* if $a_1 < a_2$ and $b_1 < b_2$, then $a_1\&b_1 < a_2\&b_2$.
(6) *Continuity:* the function $\&$ is continuous in both variables.

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Definition 2. By a *continuous, strictly increasing t-conorm*, we mean a function

$$\vee : [0,1] \times [0,1] \to [0,1]$$

that satisfies the following conditions:

(1) *Boundary conditions:* $x \vee 0 = 0 \vee x = x$, $x \vee 1 = 1 \vee x = 1$.
(2) *Commutativity:* $x \vee y = y \vee x$.
(3) *Associativity:* $x \vee (y \vee z) = (x \vee y) \vee z$.
(4) *Monotonicity:* if $a_1 \le a_2$ and $b_1 \le b_2$, then $a_1 \vee b_1 \le a_2 \vee b_2$.
(5) *Strict monotonicity:* if $a_1 < a_2$ and $b_1 < b_2$, then $a_1 \vee b_1 < a_2 \vee b_2$.
(6) *Continuity:* the function $\vee$ is continuous in both variables.

Comment. These conditions are easy to justify: e.g., commutativity means that we want to assign equal estimate to statements $A\&B$ and $B\&A$, etc.

CLASSIFICATION THEOREM FOR AND OPERATIONS. [1] *Any continuous, strictly monotonic t-norm can be represented as* $a\&b = \varphi^{-1}(\varphi(a) \cdot \varphi(b))$ *for some continuous and strictly increasing function* $\varphi : [0,1] \to [0,1]$ *for which* $\varphi(0) = 0$ *and* $\varphi(1) = 1$.

Comment. This result states that any continuous strictly increasing t-norm & is equivalent to the algebraic product.

CLASSIFICATION THEOREM FOR OR OPERATIONS. [1] *Any continuous, strictly monotonic t-conorm* $\vee$ *can be represented as* $a \vee b = \psi^{-1}(\psi(a) \cdot \psi(b))$ *for some continuous and strictly decreasing function* $\psi : [0,1] \to [0,1]$ *for which* $\psi(0) = 1$ *and* $\psi(1) = 0$.

IV. Intervals Are Necessary To Describe Degrees Of Belief

Experts cannot describe their degrees of belief precisely. At best, they can give an *interval* of possible values. For example, an expert can point to 8 on a scale from 0 to 10, but this same expert will hardly be able to pinpoint a value on a scale from 0 to 100. As a result, the only thing that we know about the expert's degree of belief is that it is closer to 8 than to 7 or to 9, i.e., that it is in the interval $[0.75, 0.85]$.

So, to describe degrees of belief more adequately, we must use *intervals* instead of real numbers. This idea have been proposed by Turksen [9] and Bandler and Kohout [10].

Since we went from numbers to intervals, we need AND and OR operations as functions from intervals to intervals. In this paper, we give a complete classification of such operations.

V. Main Results

To formulate these results, we will need the following denotations:

Denotations.

- By I, we will denote the set of all intervals $\mathbf{a} = [a^-, a^+] \subseteq [0,1]$ (i.e., all intervals for which $0 \le a^- \le a^+ \le 1$).
- We say that $\mathbf{a} = [a^-, a^+] \le [b^-, b^+] = \mathbf{b}$ if $a^+ \le b^-$.
- We say that $\mathbf{a} = [a^-, a^+] < [b^-, b^+] = \mathbf{b}$ if $a^+ < b^-$.

Comment. The notation $[a^-, a^+] < [b^-, b^+]$ means the following: if $d(A) = [a^-, a^+]$ and $d(B) = [b^-, b^+]$, then our belief in B is stronger than our belief in A. Since we do not know the exact values of these degrees of belief, the only way to express this "stronger" is to require that $a < b$ for all $a \in [a^-, a^+]$ and for all $b \in [b^-, b^+]$. This condition is equivalent to $a^+ < b^-$.

Definition 3. By an *interval AND operation*, we mean a function $\& : I \times I \to I$ that satisfies the following conditions:

(1) *Boundary conditions:* $\mathbf{x}\&[0,0] = [0,0]\&\mathbf{x} = [0,0]$, $\mathbf{x}\&[1,1] = [1,1]\&\mathbf{x} = \mathbf{x}$.
(2) *Commutativity:* $\mathbf{x}\&\mathbf{y} = \mathbf{y}\&\mathbf{x}$.
(3) *Associativity:* $\mathbf{x}\&(\mathbf{y}\&\mathbf{z}) = (\mathbf{x}\&\mathbf{y})\&\mathbf{z}$.
(4) *Monotonicity:* if $\mathbf{a}_1 \le \mathbf{a}_2$ and $\mathbf{b}_1 \le \mathbf{b}_2$, then $\mathbf{a}_1\&\mathbf{b}_1 \le \mathbf{a}_2\&\mathbf{b}_2$.
(5) *Strict monotonicity:* if $\mathbf{a}_1 < \mathbf{a}_2$ and $\mathbf{b}_1 < \mathbf{b}_2$, then $\mathbf{a}_1\&\mathbf{b}_1 < \mathbf{a}_2\&\mathbf{b}_2$.
(6) *Continuity:* the function $\mathbf{a}, \mathbf{b} \to \mathbf{a}\&\mathbf{b}$ is continuous in both variables (i.e., in all four variables a^-, a^+, b^-, b^+).
(7) *Inclusion monotonicity:* if $\mathbf{a}_1 \subseteq \mathbf{a}_2$ and $\mathbf{b}_1 \subseteq \mathbf{b}_2$, then $\mathbf{a}_1\&\mathbf{b}_1 \subseteq \mathbf{a}_2\&\mathbf{b}_2$.

Definition 4. By an *interval OR operation*, we mean a function $\vee : I \times I \to I$ that satisfies the following conditions:

(1) *Boundary conditions:* $\mathbf{x} \vee [0,0] = [0,0] \vee \mathbf{x} = \mathbf{x}$, $\mathbf{x} \vee [1,1] = [1,1] \vee \mathbf{x} = [1,1]$.
(2) *Commutativity:* $\mathbf{x} \vee \mathbf{y} = \mathbf{y} \vee \mathbf{x}$.
(3) *Associativity:* $\mathbf{x} \vee (\mathbf{y} \vee \mathbf{z}) = (\mathbf{x} \vee \mathbf{y}) \vee \mathbf{z}$.
(4) *Monotonicity:* if $\mathbf{a}_1 \le \mathbf{a}_2$ and $\mathbf{b}_1 \le \mathbf{b}_2$, then $\mathbf{a}_1 \vee \mathbf{b}_1 \le \mathbf{a}_2 \vee \mathbf{b}_2$.
(5) *Strict monotonicity:* if $\mathbf{a}_1 < \mathbf{a}_2$ and $\mathbf{b}_1 < \mathbf{b}_2$, then $\mathbf{a}_1 \vee \mathbf{b}_1 < \mathbf{a}_2 \vee \mathbf{b}_2$.
(6) *Continuity:* the function $\mathbf{a}, \mathbf{b} \to \mathbf{a} \vee \mathbf{b}$ is continuous in both variables (i.e., in all four variables a^-, a^+, b^-, b^+).
(7) *Inclusion monotonicity:* if $\mathbf{a}_1 \subseteq \mathbf{a}_2$ and $\mathbf{b}_1 \subseteq \mathbf{b}_2$, then $\mathbf{a}_1 \vee \mathbf{b}_1 \subseteq \mathbf{a}_2 \vee \mathbf{b}_2$.

Comment. The motivation for inclusion monotonicty is as follows: suppose that initially, we had $\mathbf{a}_2$ and $\mathbf{b}_2$ as sets of possible values of degrees of belief in A and B. Then, by applying the interval AND operation &, we can conclude that the degree of belief in $A\&B$ is in $\mathbf{a}_2\&\mathbf{b}_2$.

Suppose now that we have narrowed down our degrees of belief to $\mathbf{a}_1 \subseteq \mathbf{a}_2$ and $\mathbf{b}_1 \subseteq \mathbf{b}_2$. If we apply the same interval AND operation to the new degrees of belief, we get a new interval $\mathbf{a}_1\&\mathbf{b}_1$. Since we have narrowed down our intervals of possible degrees of belief, it can happen that some previously possible degrees of belief in $A\&B$ are not possible any more. But it is reasonable to require that if a value is now possible, then it was possible earlier as well (when we had even fewer knowledge about degrees of belief). In other words, we require that every number from $\mathbf{a}_1\&\mathbf{b}_1$ belongs to $\mathbf{a}_2\&\mathbf{b}_2$. This is exactly inclusion monotonicity.

CLASSIFICATION THEOREM FOR INTERVAL AND OPERATIONS. *For any interval AND operation &, there exists a continuous and strictly increasing function $\varphi : [0,1] \to [0,1]$ such that $\varphi(0) = 0$, $\varphi(1) = 1$, and*

$$[a^-, a^+]\&[b^-, b^+] =$$

$$[\varphi^{-1}(\varphi(a^-)\varphi(b^-)), \varphi^{-1}(\varphi(a^+)\varphi(b^+))].$$

CLASSIFICATION THEOREM FOR INTERVAL OR OPERATIONS. *For any interval OR operation $\vee$, there exists a continuous and strictly decreasing function $\psi : [0,1] \to [0,1]$ such that $\psi(0) = 1$, $\psi(1) = 0$, and*

$$[a^-, a^+] \vee [b^-, b^+] =$$

$$[\psi^{-1}(\psi(a^-)\psi(b^-)), \psi^{-1}(\psi(a^+)\psi(b^+))].$$

Comment. It is easy to check that the inverse if also true: if φ (or ψ) satisfies the given conditions, then the formula given in the formulations of the corresponding theorem defines an interval AND (OR) operation.

VI. Proofs

We will prove the result for AND operations. The result for OR operations is proven similarly.

Let us denote the lower and upper bounds of the interval $[a^-, a^+]\&[b^-, b^+]$ by $f^-(a^-, a^+, b^-, b^+)$ and $f^+(a^-, a^+, b^-, b^+)]$. The proof will consist of the following two steps:

- First, we will prove the following three statements:
  - f^- depends only on a^- and b^-;
  - f^+ depends only on a^+ and b^+;
  - $f^-(x, y) = f^+(x, y)$.
- Second, we will prove that f^- is a continuous, strictly increasing t-norm. The desired result will then follow from the classification theorem for non-interval AND operations.

Let us first prove the desired three statements. Since & is an interval AND operation, it is monotonic. So, from $[a^-, a^-] \le [a^-, a^+]$ and $[b^-, b^-] \le [b^-, b^+]$, we can conclude that

$$f^+(a^-, a^-, b^-, b^-) \le f^-(a^-, a^+, b^-, b^+). \quad (1)$$

Since f^- and f^+ are endpoints of an interval, we have

$$f^-(a^-, a^+, b^-, b^+) \le f^+(a^-, a^+, b^-, b^+) \quad (2)$$

and

$$f^-(a^-, a^-, b^-, b^-) \le f^+(a^-, a^-, b^-, b^-). \quad (3)$$

On the other hand, we can apply inclusion monotonicity: since $[a^-, a^-] \subseteq [a^-, a^+]$ and $[b^-, b^-] \subseteq [b^-, b^+]$, we have

$$[f^-(a^-, a^-, b^-, b^-), f^+(a^-, a^-, b^-, b^-)] \subseteq$$
$$[f^-(a^-, a^+, b^-, b^+), f^+(a^-, a^+, b^-, b^+)].$$

This interval inclusion means that

$$f^-(a^-, a^+, b^-, b^+) \le f^-(a^-, a^-, b^-, b^-) \quad (4)$$

and

$$f^+(a^-, a^+, b^-, b^+) \ge f^+(a^-, a^-, b^-, b^-). \quad (5)$$

From (3), (1), and (4), we conclude that

$$f^-(a^-, a^-, b^-, b^-) \le f^+(a^-, a^-, b^-, b^-) \le$$
$$f^-(a^-, a^+, b^-, b^+) \le f^-(a^-, a^-, b^-, b^-).$$

Therefore,

$$f^-(a^-, a^-, b^-, b^-) = f^+(a^-, a^-, b^-, b^-) =$$
$$f^-(a^-, a^+, b^-, b^+). \quad (6)$$

Since

$$f^-(a^-, a^-, b^-, b^-) = f^-(a^-, a^+, b^-, b^+),$$

we conclude that f^- depends only on a^- and b^-. Similarly, f^+ depends only on a^+ and b^+, and from (6), it follows that $f^-(x, y) = f^+(x, y)$ for all x and y.

So, for degenerate intervals $[x, x]$ and $[y, y]$, the result $[f^-(x, y), f^+(x, y)]$ of applying an interval AND operation is also a degenerate interval. Hence, we have a function $[0,1] \times [0,1] \to [0,1]$. For this function, conditions (1)–(6) lead to similar conditions in Defintion 1. So, due to Classification Theorem for AND operations, we get the desired expression for $f^- = f^+$ in terms of φ. Q.E.D.

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