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Choosing Interval Functions To Represent Measurement Inaccuracies: Group-Theoretic Approach

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I. Formulation of the Problem

A. We Need Error Estimates

Measurements are not 100% accurate. The manufacturer of a measuring instrument usually supplies us with the guaranteed upper bound Δ on the measurement error. As a result, if the measurement result is $\tilde{x}$, then we are guaranteed that the actual value x of the measured quantity lies in the *interval* $\mathbf{x} = [\tilde{x} - \Delta, \tilde{x} + \Delta]$.

The bound may be applicable for all $\tilde{x}$, or different measurement error bounds may be described for different values x of the measured quantity: $\Delta = \Delta(\tilde{x})$. In this case, we have an *interval function* $\mathbf{x}(\tilde{x}) = [\tilde{x} - \Delta(\tilde{x}), \tilde{x} + \Delta(\tilde{x})]$.

In some cases, we also know the *probabilities* of different values of error (and therefore, of different values inside the interval $\mathbf{y}$). However, in many real-life cases, we do not know these probabilities, and the interval is the only information that we have.

B. How Are Error Bounds Determined?

From the viewpoint of the user, as soon as the dependency $\Delta(\tilde{x})$ is supplied, the problem of describing the possible error is thus solved. But the manufacturer still needs to determine it, i.e., to *calibrate* his measuring instrument.

To calibrate an instrument, we must measure the same set of quantities by using two different measuring instruments:

- the calibrated one, and
- a more precise one that is used as a *standard.*

If the second instrument is much more accurate than the calibrated one, then we can neglect the measurement errors of the second instrument, and treat the results of its measurements as the actual values of x. Then, the difference between the results of measuring the same quantity by these two instruments can be treated as the measurement error of the first instrument $\Delta x = \tilde{x} - x$. For the same input value, if we repeat the measurements many times, with varying conditions, then we will get several possible values of error Δx. If we have made sufficiently many calibrating measurements, that lead to different values $\Delta x_1, ..., \Delta x_N$, then we can safely take the largest value $\max|\Delta x_i|$ as the desired estimate $\Delta(\tilde{x})$.

C. The Problem

Ideally, we should repeat this procedure for every $\tilde{x}$, and get the desired function. In reality, however, we can only do it for finitely many values $\tilde{x}_1, ..., \tilde{x}_n$. Therefore, we only get finitely many value $\Delta(\tilde{x}_1), ..., \Delta(\tilde{x}_n)$. From finitely many values, we can only determine finitely many numerical characteristics of the desired function $\Delta(\tilde{x})$. If we do not restrict our class of functions a priori, we will not be able to determine the desired interval function uniquely, because one needs *infinitely many* parameters to determine a function (in mathematical terms, the set of all functions is *infinite-dimensional*).

So, in order to complete the calibration, we need to choose a *finite-dimensional* family of interval functions. Then, we will determine its parameters from the measurement results. Therefore, we arrive at the following problem: *to choose a finite-dimensional family of interval functions.*

II. Main Idea: Describing Measurement Transformations

To choose an appropriate family of interval functions, we will use two ideas:

- First, an interval function can be viewed as a *set* of functions:
$$\mathbf{f}(x) = [f^-(x), f^+(x)] =$$
$$\{f(x) \mid f(x) \in [f^-(x), f^+(x)] \text{ for all } x\}.$$
So, to describe interval functions, we will first describe the elements functions.
- To describe these possible functions, we will take into consideration the fact that inside the measuring instrument, there usually is a sequence of *measurement transformations*: e.g.,

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from temperature, we go to current, and then from current, to numbers; usually, there are more stages (this idea was first described and used in [5]).

So, the desired function is a *composition* of several measurement transformation functions.

The class F of all possible measurement transformations must thus have the following properties:

- If a function $a \to r(a)$ is a measurement transformation (from a stage A to some other stage B), and a function $b \to s(b)$ is a measurement transformation, then we can combine these two transformation and get a measurement transformation that is equal to the composition of r and s. So, it is reasonable to demand that the class F of all measurement transformations must be closed under composition.
- Suppose that $a \to r(a)$ is a non-trivial measurement transformation. The goal of the measurement is to reconstruct the original value a. Therefore, in order to use the transformation $a \to r(a)$ in the measuring instruments, we must be able to reconstruct a from the result of the transformation. In other words, we must have an *inverse* transformation r^{-1} that transforms $r(a)$ back into a in our family F.

  Comment. Thus, the family F must contain the inverse of every function that belongs to it, and the composition of every two functions from F. In mathematical terms, it means that F must be a *transformation group.*
- We have already mentioned that we want the family of transformations F to depend on finitely many parameters. In mathematics, the notion of a group whose elements are continuously depending on finitely many parameters is formalized as the notion of a (connected) *Lie group.* So, we conclude that measurement transformations must form a connected Lie group.
- There are example of reasonable measurement transformations:
  - In many cases, we need an *amplification* of the input signal is $a \to ka$ for a constant k;
  - In many real-life situations, there is a constant additive noise. To get rid of this noise, we must *subtract* its value from the signal. The corresponding measurement transformation is $a \to a+C$ for some constant C.

  So, the set of measurement transformations must contain all functions of the type $a \to ka$ and $a \to a + C$.

So, the set F of all measurement transformations is a finite-dimensional Lie group that contains all function of the type $a \to ka$ and $a \to a + C$ (since F is a group, it will also contain all linear transformation $a \to ka + l$ for $k > 0$).

Such sets have been described in mathematics. Namely, the problem of classifying all finite-dimensional transformation groups of an n-dimensional space R^n (where $n = 1, 2, 3, ...$) that include a sufficiently big family of linear transformations, was formulated by N. Wiener (see, e.g., [15]). Wiener also formulated a hypothesis that was confirmed in [7] and [13]. It turned out that for $n = 1$, only two groups are possible:

- the group of all linear transformations and
- the group of all fractionally linear transformations
$$a \to \frac{k_1 a + l_1}{k_2 a + l_2}.$$

In both cases, all elements from F are fractionally linear. So, we can conclude that *every measurement transformation from a class F is fractionally linear.*

Comments.

1. A simplified proof of the general theorem for $n = 1$ is given in [12].
2. In [3, 4, 5], we have shown that fractionally-linear transformations are indeed a good description of measurement transformation of the actual sensors (at least locally).
3. This result can be viewed as a *limit theorem* for transformation functions [5] (similar to limit theorems from mathematical statistics citeG54,AZ88). Indeed:

- In statistics, we describe random variables that can be represented as a sum of many small components.
- Here, we want to describe transformations that can be represented as a composition of many transformations that are small in the sense that they practically do not change the value $f(x) \approx x$.

In statistics, to describe the desired class of variables, we consider probability distributions that are *infinitely divisible*, i.e., that can be represented as a sum of arbitrarily small components. Similarly, we want to describe the limit family of the transformations that can be this represented in the above-described composition form with component transformations that are arbitrarily close to 0. The classification only makes sense if we have a *finite-dimensional* family of limit transformations. It is easy to argue that this limit family must be invariant w.r.t. composition and inversion, and that it must contain linear transformations. So, this limit family must be a finite-dimensional Lie transformation group that contains linear transformations. Hence, due to the above-mentioned result, this family must consist of fractionally linear transformations only.

4. For other applications of this result, see [8, 2, 9, 10, 11].

III. Final Result: Describing Interval Functions

Now that we know how to describe *individual* measurement transformations, we can describe *sets* of these transformations, i.e., desired *interval functions.*

The main idea here is that the only information that we have about the errors are the intervals. We do not know anything about the possible correlation between the intervals that correspond to different values of $\tilde{x}$. So, if for some value $\tilde{x}$, the error takes the largest possible value $\Delta(\tilde{x})$, it is still possible that for other value of the input signal, the error will also take the largest possible value. In other words, we assume that the function $f^+(\tilde{x}) = \tilde{x} + \Delta(\tilde{x})$ is a *possible measurement transformation.*

Similar arguments lead us to a conclusion that the function $f^-(\tilde{x}) = \tilde{x} - \Delta(\tilde{x})$ (that describes the largest possible *negative* errors) is also a *possible measurement transformation.*

So, the desired interval function $\mathbf{x}(\tilde{x})$ has the form $[f^-(\tilde{x}), f^+(\tilde{x})]$, where $f^\pm$ are measurement transformations.

We already know that measurement transformations are fractionally-linear functions. So, we arrive at the following conclusion: *measurement inaccuracies must be described by the interval functions of the type* $[f^-(\tilde{x}), f^+(\tilde{x})]$, *where both functions* $f^\pm$ *are fractionally linear.*

This result is in good accordance with the experimental data about sensors.

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