A Remark on Power Series Estimation via Boundary Corrections with Parameter

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I. Formulation of the Problem

Many numerical methods are based on the fact that the desired result s can be represented as a sum of an infinite series

$$s := \sum_{n=1}^{\infty} a_n : \qquad (1)$$

Taylor series, Fourier series, etc. Such series are typical, e.g., in physical applications. Due to this representation, we can approximate the desired result s with a finite sum

$$\sum_{n=1}^{N} a_n;$$

the larger N, the better the approximation. The majority of the existing methods are *heuristic:* they either fix N from the very beginning, or choose N for which a_N becomes sufficiently small. These heuristic methods, however, do not guarantee any accuracy of the resulting estimate. It is therefore desirable to design estimates with a guaranteed estimate.

If we use a finite sum with some N as an estimate for s, then the error of this estimate can also be represented as a sum of the infinite series:

$$\sum_{n=N+1}^{\infty} a_n.$$

Therefore, if we have a method of estimating the sum of an infinite series, then this same method will enable us to estimate the error of the $N-$term approximation. One way to estimate the sum of the series is to take into consideration that a series can be viewed as a quadrature for an integral over an infinite domain:

$$I(f) = \int_1^{\infty} f(x)\,dx \qquad (2)$$

for some function $f : [1, \infty) \to R$ for which $f(n) = a_n$ for all n. Therefore, if for a given series, we know an explicitly integrable function $f(x)$ with this property, then we can take the value $I(f)$ of the integral as an estimate for s. How accurate is this estimate?

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II. Main Idea

In this paper, we describe a way to estimate the accuracy of this estimate $I(f)$. This method (based on formulas from [2, 3, 4]) will be applicable for the case when the function $f(x)$ satisfies the following properties:

- $f(x)$ is *monotonic.* Since the sum (1) converges, it means that f is either positive and decreasing, or negative and increasing;
- f must be *smooth*, namely, for some $\bar{\beta} > 0$, f is four times differentiable on the semi-line $[1-\bar{\beta}, \infty)$;
- the fourth derivative $f^{IV}(x)$ of the function f must be different from 0 for all x from this semi-line; and
- the third derivative $f^{III}(x)$ must have a finite limit as $x \to \infty$.

III. Main Result

If $f^{IV}(x) > 0$ for all $x \in [1-\bar{\beta}, \infty)$, then for every $\bar{\beta} \in (0, 1]$ and for every $\beta \in [\sqrt{6}/8, 1]$, we get the following estimates:

$$\frac{a_1}{2} + \int_1^{\infty} f(x)\,dx -$$

$$\frac{1}{24\beta}[-3f(1) + 4f(1+\beta) - f(1+2\beta)] \le s \le$$

$$\frac{a_1}{2} + \int_1^{\infty} f(x)\,dx - \frac{1}{24\bar{\beta}}[f(1+\bar{\beta}) - f(1-\bar{\beta})]. \qquad (3)$$

If $f^{IV}(x) < 0$, then we get a similar inequality, but with the right-hand side instead of the left-hand side and vice versa.

IV. Proof

The desired formula (3) follows from the following two quadrature formulas:

The first formula was proven in [2, 3]:

$$\int_a^b f(x)\,dx = T_{N+1}[f] +$$

$$\frac{h}{24\bar\beta}[f(a+\bar\beta h) - f(a-\bar\beta h) - f(b+\bar\beta h) + f(b-\bar\beta h)]$$

$$+R^{\bar\beta}_{N+5}[f],$$

where

$$R^{\bar\beta}_{N+5}[f] = \frac{1+10\bar\beta^2}{720}(b-a)h^4 f^{IV}(\xi)$$

for some $\xi \in (a,b)$.

The second formula was proven in [3, 4]:

$$\int_a^b f(x)\,dx = T_{N+1}[f] +$$

$$\frac{h}{24\beta}[-3f(a) + 4f(a+\beta h) - f(a+2\beta h) - 3f(b) +$$

$$4f(b-\beta h) - f(b-2\beta h)]$$

$$+\tilde R^{\beta}_{N+5}[f],$$

where

$$\tilde R^{\beta}_{N+5}[f] = \frac{h^5}{720}[30\beta^3 + N(1-20\beta^2)]f^{IV}(\delta)$$

for some $\delta \in (a,b)$ (this formula is only true if $\beta \in [\sqrt{6}/8, 1]$).

We apply these formulas for $a = 1$, $b = N$, and $h = 1$. Then, the integral sum $T_{N+1}(f)$ is equal to $a_1 + a_2 + \ldots + a_N$.

For our choice of β, we have $1 - 20\beta^2 \le 0$. Therefore, if the fourth derivative is always positive, then $R \ge 0$ and (for sufficiently large N), $\tilde R \le 0$. So, if we let N go to ∞, and take into consideration that $f(x) \to 0$ as $x \to \infty$, we get the desired inequalities.

If $f^{IV} < 0$, then we get the opposite inequalities.

Comment. We can also get the explicit expressions for the difference between s and its estimates from (3). Namely, the Peano kernels $K_{\dots}^{\dots}$ of these quadrature formulas are of opposite signs, and $f^{IV} \ne 0$. Therefore, we can apply the mean value theorem to the integrals that describe the remainders of these formulas:

$$R^{\bar\beta}_{N+5}[f] = \int_{a-\bar\beta h}^{b+\bar\beta h} K_4^{\bar\beta}(x)\,dx =$$

$$= K_4^{\bar\beta}(\alpha)[f^{III}(b+\bar\beta h) - f^{III}(a-\bar\beta h)]$$

for some $\alpha \in (a-\bar\beta h, b+\bar\beta h)$. Similarly,

$$\tilde R^{\beta}_{N+5}[f] = \int_a^b \tilde K_4^{\beta}(x)\,dx =$$

$$= \tilde K_4^{\bar\beta}(\gamma)[f^{III}(b) - f^{III}(a)]$$

for some $\gamma \in (a,b)$. When $N \to \infty$, these terms turn into

$$R^{\bar\beta}_\infty[f] = \frac{\nu}{24}\left(\frac{1}{16} + \frac{\bar\beta^2}{3}\right)[f^{III}(1-\bar\beta) - \lim_{x\to\infty} f^{III}(x)] \tag{4}$$

for some $\nu \in [0,1]$, and

$$\tilde R^{\bar\beta}_\infty[f] = -\mu\frac{\beta^2}{36}[f^{III}(1) - \lim_{x\to\infty} f^{III}(x)] \tag{5}$$

for some $\mu \in [0,1]$. These terms are always non-positive and non-negative. Therefore, in the limit $N \to \infty$, the above-described quadrature formulas lead to the desired inequalities.

V. Examples

A. Example 1

Let us estimate the value $\zeta(3)$ of the Riemann zeta function

$$\zeta(3) = \sum_{n=1}^{\infty}\frac{1}{n^3}$$

(cf. [1], p. 232). As a first approximation to the sum, we will take the sum of its first nine terms. Then, to get the estimate for the sum itself, we must estimate the remainder

$$s = \sum_{n=10}^{\infty}\frac{1}{n^3} = \sum_{n=1}^{\infty}\frac{1}{(9+n)^3}.$$

From (3), we conclude that

$$\frac{1}{2\cdot 10^3} + \int_1^\infty \frac{dx}{(9+x)^3} -$$

$$\frac{1}{24\beta}\left[-\frac{3}{10^3} + \frac{4}{(10+\beta)^3} - \frac{1}{(10+2\beta)^3}\right] \le s \le$$

$$\frac{1}{2\cdot 10^3} + \int_1^\infty \frac{dx}{(9+x)^3} - \frac{1}{24\bar\beta}\left[\frac{1}{10+\bar\beta)^3} - \frac{1}{10-\bar\beta)^3}\right].$$

If we choose $\beta = 0.5$ and $\bar\beta = 0.1$, the for

$$\zeta(3) = s + \sum_{n=1}^{9}\frac{1}{n^3}$$

we get an estimate

$$1.2020567 \le \zeta(3) \le 1.2020571,$$

i.e., an estimate with an accuracy $\le 4\cdot 10^{-7}$ (actually, in this case,

$$\tilde R^{\beta}_\infty = \mu\cdot 4.16\cdot 10^{-7}$$

and

$$R^{\bar\beta}_\infty = -\nu\cdot 1.65\cdot 10^{-7}.)$$

B. Example 2

Estimate

$$s = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \tag{6}$$

(cf. [1], p. 243). In this case, the values a_n are not monotonic. So, to apply our method, we must represent the sum of this series in the desired form. This can be done by combining together neighboring terms of opposite sign.

We can start this combination with the very first term, and thus get

$$s = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \big(-\frac{1}{\sqrt{2n-1}} + \frac{1}{\sqrt{2n}} \big),$$

so that

$$a_n = -\frac{1}{\sqrt{2n-1}} + \frac{1}{\sqrt{2n}},$$

$$f(x) = -\frac{1}{\sqrt{2x-1}} + \frac{1}{\sqrt{2x}},$$

and

$$f^{IV}(x) = -3 \cdot 5 \cdot 7 \cdot (2x-1)^{-9/2} + 3 \cdot 5 \cdot 7 \cdot (2x)^{-9/2} > 0$$

for all $x > 1/2$.

We can also start combining with the second term, in which case we end up with

$$s = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} = -1 + \sum_{n=1}^{\infty} \big(\frac{1}{\sqrt{2n}} - \frac{1}{\sqrt{2n+1}} \big),$$

so that

$$a_n = \frac{1}{\sqrt{2n}} - \frac{1}{\sqrt{2n+1}},$$

$$f(x) = \frac{1}{\sqrt{2x}} - \frac{1}{\sqrt{2x+1}},$$

and $f^{IV}(x) = 3 \cdot 5 \cdot 7 \cdot (2x)^{-9/2} - 3 \cdot 5 \cdot 7 \cdot (2x+1)^{-9/2} > 0$ for all $x > 0$.

Applying an estimate with $\bar{\beta} = 0.01$ to both formulas, we get

$$-0.6164982 \le s \le -0.6037183.$$

Applying an estimate with $\beta = 0.5$, we get

$$-0.6048987 \le s \le -0.59707716.$$

Combining these two inequalities, we get the following estimate for s:

$$-0.6048987 \le s \le -0.6037183.$$

To get a better estimate, we compute the first nine terms separately, and estimate the remainder:

$$s = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} = \sum_{n=1}^{9} \frac{(-1)^n}{\sqrt{n}} + \sum_{n=5}^{\infty} \big(\frac{1}{\sqrt{2n}} - \frac{1}{\sqrt{2n+1}} \big).$$

For the remaining sum, for $\bar{\beta} = 0.1$ and $\beta = \sqrt{6}/8$, we get

$$\tilde{R}^{\beta}_{\infty} = \mu \cdot 1.4304291 \cdot 10^{-6}$$

and

$$R^{\bar{\beta}}_{\infty} = -\nu \cdot 3.690692 \cdot 10^{-6}.$$

References

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