In Case of Interval Uncertainty, Optimal Control is NP-Hard Even for Linear Plants, so Expert Knowledge is Needed

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***Abstract*— If we know the plant precisely, then in the first approximation (for a linear plant and quadratic objective function), the problem of optimal control has an explicit solution. In this paper, we show that if we take into consideration the fact that in real life, all parameters of a plant are known with interval uncertainty, then even in the first approximation, the problem of optimal control becomes computationally intractable (NP-hard). This result justifies the necessity to use expert knowledge in control (and corresponding intelligent control methodologies).**

I. Mathematical Optimal Control Theory: A Brief General Formulation of a Problem

Traditional control theory (see, e.g., [5]) deals with the case when our knowledge is complete:

- We know exactly the initial state $x(0) = x_0$ of the plant (i.e., of the object that we want to control). The state of a plant is usually described by its characteristics $x_1, ..., x_n$; so, from mathematical viewpoint, the state is an $n-$dimensional vector $x = (x_1, ..., x_n)$.
- We know exactly how the state of the plant changes if we apply different controls u. In precise mathematical terms, we know the differential equation $\dot{x} = f(x, u)$ that describes the plant.
- We know what exactly we want to achieve by this control, i.e., we know a functional $J(x, u)$ (called *objective function*) that we want to minimize (or maximize) by applying the optimal control. For example, if we are planning a flight to the planet:
  - we may want to minimize the time that is necessary to reach that planet; this objective function depends only on the trajectory $x(t)$);
  - or, we may want to minimize the total amount of fuel spent during this flight; this objective function depends only on the control $u(t)$;
  - we may also want to minimize some combination of time and fuel spent; in this case, we have an objective function that depends on both x and u.

In this case, the problem of finding the optimal control becomes a well-defined mathematical problem:

$$J(x, u) \to \min_u$$

under the conditions that $x(0) = x_0$ and $\dot{x} = f(x, u)$.

II. This Problem Has Been Solved in the First Approximation

In many important cases, this optimization problem has been solved, and we have explicit formulas for optimal control that are being successfully used in many real-life applications. Of course, there are still many complicated problems, with complicated evolution function f and complicated objective function J, for which an optimization technique still need to be found, but at least in the *first approximation*, the problem is solved. In order to describe this result in mathematical terms, we must describe what a "first approximation" means in this case.

Usually, we consider the plants for which both the evolution function f and the objective function J depend *smoothly* (i.e., differentially) on both x and u. Of course, one can imagine realistic situations in which the dependency is not smooth and even not continuous at all: e.g., in sports-like competition situations, we either score or we do not; we either hit the planet, or we do not. In such situations, the payoff function $J(x, u)$ takes only two values: 0 and 1, and it therefore, cannot be continuous. Such situations are, however, an idealization. In real-life situations, e.g., in spaceship trajectory control, it is not true that when we change the thrust continuously, we at some point immediately switch from

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hitting the planet to not hitting it: due to inevitable uncertainties, we can only talk about the *probability* of a hit, and this probability smoothly changes with x and u. A similar argument can be applied to other situations, so, we can safely assume that both the evolution function and the objective function are smooth.

In this case, the notion of a first approximation is very natural: by definition, a smooth function is the one that can be (at least locally) well approximated by a linear, quadratic, etc functions (depending on how many derivatives we require). So, we can take the first significant term in the Taylor expansion as the first approximation of the function.

- For an evolution function $f(x,u)$, we cannot take the 0-th order (constant) term, because then, the system will be not controllable at all. So, to get a reasonable first approximation, we must also retain *linear* terms in f. As a result, we get a first approximation of the type $\dot{x} = a + Ax + Bu$ for some vector a and matrices A and B.
- For the objective function, linear approximation is not enough, because linear functions do not have any minima at all. So, we also need to consider terms that are quadratic in $x(t)$ and $u(t)$. In other words, we must consider objective functions of the type

$$J(x,u) = J_0 + \int J_x(t)x(t)\,dt + \int J_u(t)u(t)\,dt +$$
$$\int J_{xx}(t)x(t)x(t)\,dt + \int J_{xu}x(t)u(t)\,dt +$$
$$\int J_{uu}(t)u(t)u(t)\,dt$$

  for some constant J_0, vectors $J_x(t)$ and $J_u(t)$ and matrices $J_{xx}(t)$, $J_{xu}(t)$, and $J_{uu}(t)$. The constant term can be safely dropped, because if J_0 is a constant, then $J \to \min$ iff $J - J_0 \to \min$.

For linear systems, and quadratic objective functions, there exists an explicit solution to the optimal control problem.

It therefore seems (at first glance) that in the first approximation, the problem of optimal control is solved. We will show that the optimal control problem is more complicated than that. The reason is that in the above formulations, we were assuming that we know the plant *precisely*, but:

III. In Real Life, We Never Know Any Values Precisely

In real-life situations, we never know the precise values of any parameters that describe a physical object: we get all these values from measurements (direct or indirect), and measurement are never absolutely precise. In this paper, we are going to show that *if we take that imprecision into consideration, then even for the above-described first approximation, the optimal control problem stops being simple and becomes computationally intractable (NP-hard).* We will show that this problem is intractable even in the *static* case, when all the parameters $J_{\ldots}$ of an objective function do not depend on time at all.

Moreover, we will show that not only the problem of finding an *optimal* control (with $J \to \min$) is difficult, but even a simpler problem of finding a *good enough* control (i.e., a control for which $J \le J_g$ for some J_g) is also intractable.

To describe this result in mathematical terms, we must describe two things:

- First, we must describe uncertainty in mathematical terms.
- Second, we must describe a reasonable modification of the above-described optimization problem that would take this uncertainty into consideration.

Let us first describe uncertainty. The manufacturer of the measuring instrument must provide us with the error bounds. If no error bounds are provided, then the results of measuring by this instrument are of no use: if, e.g., we get a temperature 100, but we do not know whether the accuracy is ± 1 or ± 1000, then we cannot conclude anything about the actual temperature. In addition to this error bound, some instruments may have probabilities of different errors known, but in all cases, we must have this error bound. So, as a general case, we can consider the case when this error bound is known. In this case, whatever physical quantity we are interested in, after measuring it and getting the measurement result $\tilde{x}$, the only thing that we can guarantee about the actual value x of this quantity is that it lies in the interval $[\tilde{x} - \Delta, \tilde{x} + \Delta]$, where Δ is the error bound provided by the manufacturer of the measuring instrument. In other words, we can assume that for each parameter, we only know the *interval* of its possible values. In other words, we have an *interval uncertainty*.

If there is an interval uncertainty in the parameters of the evolution function, then, even if the initial state $x(0)$, the control $u(t)$, and the objective function J are precisely known, we cannot predict exactly what a trajectory can be, and we cannot therefore predict exactly what the value of the objective function will be, and whether the resulting control will be good enough or not.

In this case, it make sense to look for a control that *can* be good enough. For example, if we are planning an automatic mission to another planet, then we are well aware of the uncertainties, and we understand very well that there is always a potential for a failure, We want, however, to choose a control in such a way that there is also a potential for a success (and thus, to avoid the disastrous controls that leave us with no chances of succeeding). In other words, we want to find a control $u(t)$ for which at least one possible trajectory satisfies the property $J \leq J_g$.

Now, we are ready for the mathematical formulation:

IV. Mathematical Formulation of the Problem and the Main Result

Definition 1. By a *control situation with interval uncertainty*, we mean the tuple

$$(n, m, x_0, \mathbf{a}, \mathbf{A}, \mathbf{B}, J_x, J_u, J_{xx}, J_{xu}, J_{uu}, J_g, T),$$

where:

- n and m are positive integers;
- x_0 is an $n-$dimensional vector;
- $\mathbf{a}$ is an $n-$dimensional interval vector;
- $\mathbf{A}$ is an $n \times n$ interval matrix;
- $\mathbf{B}$ is an $n \times m$ interval matrix;
- J_x is an $n-$dimensional vector;
- J_u is an $m-$dimensional vector;
- J_{xx} is an $n \times n$ matrix;
- J_{xu} is an $n \times m$ matrix;
- J_{uu} is an $m \times m$ matrix;
- J_g is a real number;
- T is a positive real number (called *time*).

Definition 2. Let a control situation with interval uncertainty be given.

- By a *control*, we mean a function $u : [0, T] \to R^m$.
- Let a control $u(t)$ be given. We say that $x(t)$ is a *possible trajectory*, if x is a solution of the differential equation $\dot{x} = a + Ax + Bu$, $x(0) = x_0$ for some vector $a \in \mathbf{a}$ and matrices $A \in \mathbf{A}$ and $B \in \mathbf{B}$.
- We say that x is a *good trajectory* if it is a possible trajectory, and $J(x, u) \leq J_g$, where $J(x, u) = \int_0^T K\,dt$, and
$$K = J_x x(t) + J_u u(t) +$$
$$J_{xx} x(t) x(t) + J_{xu} x(t) u(t) + J_{uu} u(t) u(t).$$
- We say that $u(t)$ is a *good control* if for this control, there exists at least one good trajectory.

Definition 3. By an *optimal control problem in case of interval uncertainty*, we mean the following problem:

Given: a control situation with interval uncertainty.
To compute: a good control (or, if there is no good control, to return the corresponding message).

THEOREM. *Optimal control problem in case of interval uncertainty is intractable (NP-hard).*

Comment. This result is in good accordance with the similar results on NP-hardness:

- the problem of finding a *robust stable* control is in general NP-hard (for references, see [2]);
- for general non-linear systems, the problem of optimal control is NP-hard even if there is no interval uncertainty [1], etc.

V. Proof

By definition, a problem is NP-hard if any algorithm that can solve it in polynomial time (i.e., in time that is bounded by some polynomial of the input) can be transformed into an algorithm that solves *all* problems (from a certain very general class NP) also in polynomial time. To prove that our problem is NP-hard, we will reduce it to another problem that is known to be NP-hard: namely, to the following problem [3]:

Given: an interval linear system of equations $\mathbf{B}u = \mathbf{b}$.
To find: the vector u that is a *possible solution*, i.e., for which $Bu = b$ for some $B \in \mathbf{B}$ and $b \in \mathbf{b}$.

We will show that if we would be able to solve the above-formulated control problem (optimal control problem in case of interval uncertainty) in polynomial time, then we would be able to solve this interval algebraic problem in polynomial time as well. Indeed, suppose that we do have a polynomial-time algorithm $\mathcal{U}$ for solving the interval control problem, and that we are given the interval matrix $\mathbf{B}$ of dimensions $n \times m$ and the interval vector $\mathbf{b}$. Then, we can solve the problem of finding the appropriate u as follows:

- Formulate the following control problem: Take the same n, m, and $\mathbf{B}$ as in the algebraic problem, $\mathbf{a} = -\mathbf{b}$, $x_0 = 0$, $\mathbf{A} = 0$, $J_{xx} = 1$, and $J_x = J_u = J_{xu} = J_{uu} = J_g = 0$.
- Apply the algorithm $\mathcal{U}$ to this problem.
- The result $u(t)$ is the desired solution of the algebraic interval problem.

Indeed, for this system, $J(x, u) = \int x^2(t)\,dt$, so, the only way to have a good trajectory x, i.e., to have $J \leq J_g = 0$, is to have $x(t) = 0$ for all t. By definition of a possible trajectory, this means that $Bu + a = 0$, i.e., that $Bu = b$ for some $B \in \mathbf{B}$ and $b \in \mathbf{b}$. This is exactly the definition of a possible solution. Q.E.D.

VI. Conclusion

The fact that in a realistic setting (of interval uncertainty), the problem of finding the optimal control is intractable even in the first approximation, means that no general algorithm is possible that would find the optimal control for all possible plants.

Instead of this non-existence general algorithm, we must use the *experience* of expert controllers. There already exists a methodology (called *intelligent control*) that translates the experience of an expert controller into the actual control strategy (see, e.g., [4, 6]). Our result shows that *expert knowledge and intelligent control methodology are needed for optimal control*, even for the simplest case when the plant is described by a linear equation, and the objective function is quadratic.

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