Error Estimate of the Result of Measuring Laser Beam Diameter

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***Abstract*— One of the most wide-spread methods of measuring the laser beam diameter consists of placing a matrix of photoelements on its way, measuring the energy (power) in each of the elements and computing the diameter. Errors in energy measurement lead to the errors in diameter. In the present paper we estimate the resulting error of diameter measurements.**

I. Introduction

A. Why is Knowing the Diameter of a Laser Beam so important

One of the characteristics of the lasers that makes them useful in many applications is the fact that their beam is usually very thin. This helps:

- in *communication*, when we need the signal to go only in one direction and do not want to waste energy on the broad beams;
- in *microsurgery* and *microtechnology*, where we need to apply the laser beam to some small region without altering anything in the surrounding regions, etc.

In view of that, the diameter of the laser beam is one of its main characteristics.

B. Definition of the Diameter

Of course, no matter how narrow the laser beam is, there is always some dispersion of the laser light to the atmosphere and deviation due to the initial irregularities in its orientation (without such dispersion we could not see the beam if it were not oriented directly towards our eyes). However, this dispersion is relatively small and the borders of the laser beam are rather sharp. In other words, the main part of the energy of the laser beam is located in some narrow region with a sharp boundary, and so it is quite reasonable to speak about its diameter. The natural definition of this diameter is the *diameter of the circle* (orthogonal to the laser beam) *that contains the prevailing part p_0 of the total energy of this beam.* This value p_0 has to be fixed (90% or 95% or 99%).

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C. How to Measure This Diameter

The natural way to measure this diameter is:

- to place the dense matrix of photoelements in the way of the laser beam;
- to measure the intensity of light of every element of this matrix;
- to compute the total energy I of the laser beam by adding these intensities;
- to find a circle with the smallest radius r for which the sum $I(r)$ of the intensities inside this circle is greater than or equal to p_0 part of the total energy (i.e., $I(r) \geq p_0 \cdot I$).

The diameter of this circle is the desired diameter of the laser beam. This method is widely used, and it is even accepted as a National Standard in some countries [5].

D. Measurement Errors

Errors in this indirect process of measuring the diameter d are due to two reasons:

- first, the matrix may not be sufficiently dense, and
- second, the photoelements can measure the intensity only with some accuracy.

Experimental analysis [4] has shown that if the matrix is sufficiently dense, than the prevailing source of errors is the errors in measuring intensity. The probabilistic distribution of these intensity measurement errors is usually unknown, the only thing that is known is the upper bound on this error. So if the measurement result is $\tilde{I}_i$ and the error bound is Δ_i, then the actual value of the measured intensity can take any value from $\tilde{I}_i - \Delta_i$ to $\tilde{I}_i + \Delta_i$. Thus we come to the following main problem:

E. The Main Problem in Error Estimation

Given $\tilde{I}_i$ and Δ_i, how do we estimate the error in the diameter? The existing methods of estimating this error (see, e.g., [5]) give the upper bounds for the diameter error that are 2-3 times greater than the actual errors [4], so new methods are needed.

There exist several methods in numerical mathematics that are aimed at solving these kind of problems: given the error estimates of the input data of the algorithm, to give an estimate for the output data (see, e.g., the survey volume [1]), but these methods are mainly applicable to the case when this algorithm implements some differentiable function, and here the transformation from intensities to diameter is not differentiable. So it is necessary to invent new methods.

In the present paper we present such a method and estimate its computational complexity. The results were announced in [3].

II. Mathematical Formulation of the Problem and Description of the Solution

Preliminary remark. As we have already mentioned, in our computations, the only information that we need about the position of each element is its distance from the center. Therefore, in order to simplify the following denotations and computations, let's order the elements in such a way that their distance from the center does not decrease: the distance r_1 between the first element and the center is less than or equal to the distance r_2 between the second element and the center; r_2 is less than or equal to r_3 etc. So, we arrive at the following definition.

Definition 1. (*measurements and measured values*).

- By a *measurement procedure*, we mean a quadruple $(N, \vec{r}, \vec{\Delta}, p_0)$, where N is an integer, $\vec{r} = (r_1, r_2, ..., r_N)$ is a nondecreasing sequence of nonnegative real numbers, $\vec{\Delta} = (\Delta_1, \Delta_2, ..., \Delta_N)$ is a sequence of positive real numbers and p_0 is a real number from [0,1].
  - The integer N is called the *number of elements* in the matrix.
  - The value r_i is called the *distance* between $i-$th element and the center, or, for short, the *i-th distance.*
  - The value Δ_i is called the *biggest possible error* of the i-th measurement.
- By *measured intensities*, we mean a sequence of N nonnegative real numbers $\tilde{I}_1, \tilde{I}_2, ..., \tilde{I}_N$. The value $\tilde{I}_i$ is called *$i-$th measured intensity.*
- By the *total measured intensity* $\tilde{I}$, we mean the sum of all the measured intensities $\tilde{I}_1 + \tilde{I}_2 + ... + \tilde{I}_N$.
- By the *measured intensity inside the radius* r (denoted by $\tilde{I}(r)$) we mean the sum of all measured intensities $\tilde{I}_i$, for which the corresponding distance is smaller than or equal to r.
- By the *measured diameter*, we mean the smallest value $\tilde{d}$, for which $\tilde{I}(\tilde{d}/2)$ is greater than or equal to $p_0 \cdot \tilde{I}$ (or, in other words, the measured intensity inside the radius $\tilde{d}/2$ comprises the prevailing part of the total measured intensity).

Definition 2. (*possible actual values of intensity and possible errors*)

- We say that the real numbers $I_1, I_2, ..., I_N$ are *possible values of intensities* iff for every i, $|\tilde{I}_i - I_i| \leq \Delta_i$.
- By a *total intensity* for values $I_1, I_2, ..., I_N$ we mean the sum $I = I_1 + ... + I_N$.
- By a *total intensity inside the radius* r for the values $I_1, I_2, ..., I_N$ we mean the sum $I(r)$ of all the intensities I_i for all i such that r_i is smaller than or equal to r.
- By a *diameter*, corresponding to values $I_1, I_2, ..., I_N$, we mean the smallest value d, for which $I(d/2)$ is greater than or equal to $p_0 \cdot I$.
- We say that a real number d is a *possible value of the diameter*, if it is a diameter corresponding to some possible values of intensity.
- By a *possible error in measuring diameter*, we mean the absolute value of the difference $|d - \tilde{d}|$ between the measured $\tilde{d}$ and a possible value of the diameter.
- By a *biggest possible error in measuring diameter*, we mean the greatest difference $|d - \tilde{d}|$ between the measured diameter and the possible value of the diameter. This biggest possible error will be denoted by Δ.

MAIN PROBLEM.

- *Given:*
  - a measurement procedure
    $(N; r_1, r_2, ..., r_N; \Delta_1, \Delta_2, ..., \Delta_N; p_0)$ and
  - the measured intensities $\tilde{I}_1, \tilde{I}_2, ..., \tilde{I}_N$.
- *To compute: the biggest possible error Δ in measuring diameter.*

Comment. The problem is not only to compute this Δ, but to try to compute it quickly (without spending much time on computations). The ideal situation is when the necessity to make computations (i.e., to process measurement results) does not slow down the measuring devices. Thus, the running time of the ideal processing algorithm must be of the same order as the time that is necessary to input the measured values. There are N values, so the running time in this ideal case must be proportional to N.

To express this formally, let's recall the relevant definition.

Definition. By a *linear-time* algorithm, we mean an algorithm that finishes all the computations in time not exceeding cN, where N is the length of the input (in our case, the number of measurements) and c is a constant (independent on N).

THEOREM. *There exists a linear-time algorithm for computing* Δ.

III. Algorithm

1) Preliminary steps:
   a) If there exist elements $i_1, ..., i_k$ for which the distances r_i are equal: $r_{i_1} = ... = r_{i_k}$, then we replace each such group by a single element i; to this element, we ascribe the measured intensity that is equal to the sum of the correspondent intensities $\tilde{I}_i = \tilde{I}_{i_1} + ... + \tilde{I}_{i_k}$, and the error Δ_i that is equal to the sum of the corresponding errors: $\Delta_i = \Delta_{i_1} + ... + \Delta_{i_k}$. After performing this step, we have a new problem in which all the values r_i are different.
   b) Compute $q_0 = p_0/(1-p_0)$;
2) For $k = 1, 2, ..., N$, compute the values I_k^+ and I_k^- as follows:
   - $I_1^+ = \tilde{I}_1 + \Delta_1$;
   - $I_{k+1}^+ = I_k^+ + (\tilde{I}_{k+1} + \Delta_{k+1})$;
   - $I_1^- = \max\{\tilde{I}_1 - \Delta_1, 0\}$;
   - $I_{k+1}^- = I_k^- + \max\{\tilde{I}_{k+1} - \Delta_{k+1}, 0\}$.

   Actually, we are computing the largest and the smallest possible values of $I(r_i)$.
3) For $k = 1, 2, ..., N$, compute the values
   - $q_k^+ = I_k^+/(I_N^- - I_k^-)$ and
   - $q_k^- = I_k^-/(I_N^+ - I_k^+)$.

   When for the first time q_k^+ becomes greater than or equal to q_0, denote the corresponding r_k by r^+. When for the first time q_k^- becomes greater than or equal to q_0, denote the corresponding r_k by r^-.
4) Compute $\Delta = \max\{|\tilde{d} - 2r^+|, |\tilde{d} - 2r^-|\}$.

IV. Example

To illustrate this algorithm, let's consider the simplified example with 9 elements (matrix 3×3) placed in a quadratic grid; if we order them in nondecreasing order of their distances, then we get $r_1 = 0, r_2 = r_3 = r_4 = r_5 = 1$, $r_6 = r_7 = r_8 = r_9 = \sqrt{2} \approx 1.41$:

$$\begin{matrix} 6 & 3 & 7 \\ 2 & 1 & 4 \\ 8 & 5 & 9 \end{matrix}$$

Assume that the measured intensities are:

i	1	2	3	4	5	6	7	8	9
$\tilde{I}_i$	10	6	7	8	2	1	0	0	0

the maximum possible error of measuring intensities is $\Delta_i = 1$, and $p_0 = 0.9$.

Let's first find the measured diameter. In this case, the total intensity $\tilde{I}$ is equal to 34; $\tilde{I}(0) = \tilde{I}_1 = 10$ (smaller than $p_0 \cdot 34$); $\tilde{I}(1) = \tilde{I}_1 + \tilde{I}_2 + ... + \tilde{I}_5 = 33$; this value is already greater than $p_0 \cdot 34$; so, the measured diameter $\tilde{d}$ is equal to $2 \cdot 1 = 2$.

Let's now apply our algorithm to compute the maximal possible error Δ of this value.

1) a) In this case, there exist elements with equal distances, so according to the first step of the proposed algorithm we must combine the elements 2 through 5 and 6 through 9 into single elements, with measured intensities and errors equal to the sum of the corresponding values. So we come to the new problem with 3 elements corresponding to old #1, to old ## 2,...,5 and to old ## 6,...,9. The new intensities are correspondingly 10, 6+7+8+2=23 and 1+0+0+0=1, the errors are correspondingly 1, 1+1+1+1=4 and 1+1+1+1=4. So the new simplified problem has the following input: $\tilde{I}_1 = 10$, $\tilde{I}_2 = 23$, $\tilde{I}_3 = 1$, $\Delta_1 = 1$, $\Delta_2 = 4$, $\Delta_3 = 4$.
   b) $q_0 = p_0/(1-p_0) = 0.9/0.1 = 9$.
2) Let's compute I_k^+ and I_k^-. Following the algorithm, we get:
   - $I_1^+ = 10 + 1 = 11$;
   - $I_2^+ = 11 + (23 + 4) = 38$;
   - $I_3^+ = 38 + (1 + 4) = 43$;
   - $I_1^- = 10 - 1 = 9$;
   - $I_2^- = 9 + (23 - 4) = 28$;
   - $I_3^- = 28 + \max(1 - 4, 0) = 28 + 0 = 28$.
3) Compute the values q_k:
   - $q_1^+ = 11/(28 - 9) < q_0 = 9$;
   - $q_1^- = 9/(43 - 11) < 9$;
   - $q_2^+ = 38/(28 - 28) = \infty$, which is greater than 9; so $r^+ = r_2 = 1$.
   - $q_2^- = 28/(43 - 38) < 9$;

- $q_3^- = 28/0 = \infty > 9$ therefore, $r^- = r_3 = 1.41$.

4) Compute $\Delta = \max\{|2-2|, |2-2.82|\} \approx 0.82$.

V. The Proof

The idea of this algorithm is very natural: the lowest possible value of the diameter corresponds to the case when the intensities inside this diameter take maximal possible values, and intensities outside take lowest possible values. Correspondingly, the lowest values outside and the biggest values outside correspond to the case, when the estimate for the diameter is lowest possible.

Let us use this idea to create a formal proof that our algorithm is correct. We will prove the following four auxiliary statements:

- If d is a possible value of a diameter, then $d \geq 2r^+$;
- If d is a possible value of a diameter, then $d \leq 2r^-$;
- $2r^+$ is a possible value of a diameter;
- $2r^-$ is a possible value of a diameter.

Because of the third and the fourth statements, the values $|\tilde{d}-2r^-|$ and $|\tilde{d}-2r^+|$ are the possible errors in measuring diameter. Therefore, the maximum $\max\{|\tilde{d}-2r^+|, |\tilde{d}-2r^-|\}$ of these two values is also a possible error in measuring diameter. Due to the first and the second statements, every possible error in measuring diameter cannot exceed this number. Therefore, this number is indeed equal to the desired biggest possible error in measuring diameter.

Let us now prove the above four statements.

- Let d be a possible value of the diameter. This means that $d = 2r_k$, where k is the first value for which

  $$I_1 + ... + I_k \geq p_0 \cdot I = p_0(I_1 + ... + I_N). \quad (1)$$

  The sum in the right-hand side of (1) can be represented as sum of two terms:

  $$I_1 + ... + I_N = (I_1 + ... + I_k) + (I_{k+1} + ... + I_N).$$

  Therefore, the inequality (1) is equivalent to

  $$I_1+...+I_k \geq p_0(I_1+...+I_k)+p_0(I_{k+1}+...+I_N).$$

  If we move the first term in the right-hand side to the left-hand side, we will get the equivalent inequality

  $$(1-p_0)(I_1 + ... + I_k) \geq p_0(I_{k+1} + ... + I_N).$$

  Dividing both sides of this inequality by $1-p_0$, we conclude that

  $$I_1 + ... + I_k \geq q_0(I_{k+1} + ... + I_N) \qquad (2).$$

  Since I_i is a possible value of i–th intensity, we have $|I_i - \tilde{I}_i| \leq \Delta_i$. Therefore, $\tilde{I}_i - \Delta_i \leq I_i \leq \tilde{I}_i + \Delta_i$. Since $I_i \geq 0$, we conclude that $I_i \geq \max(0, \tilde{I}_i - \Delta_i)$. So,

  $$I_k^+ = (\tilde{I}_1+\Delta_1)+...(\tilde{I}_k+\Delta_k) \geq I_1+...+I_k \quad (3)$$

  and

  $$I_{k+1} + ... + I_N \geq$$
  $$\max(0, \tilde{I}_{k+1} - \Delta_{k+1}) + ... + \max(0, \tilde{I}_N - \Delta_N) =$$
  $$I_N^- - I_k^-. \qquad (4)$$

  From (2), (3), and (4), we conclude that

  $$I_k^+ \geq q_0(I_N^- - I_k^-).$$

  If we divide both sides of this inequality by

  $$I_N^- - I_k^-,$$

  we conclude that $q_k^+ \geq q_0$. Since r^+ is by definition the smallest of the distances r_i for which $q_i^+ \geq q_0$, we can conclude that $r_k \geq r_i = r^+$, i.e., that $d = 2r_k \geq 2r^+$. The first statement is proven.

- Let us now prove the second statement. Let d be a possible value of the diameter. We have already proved that this means that $d = 2r_k$, where k is the first (smallest) value for which the inequality (2) holds. This, in turn, means that the inequality (2) is not true for $k-1$, i.e., that:

  $$I_1 + ... + I_{k-1} < q_0(I_k + ... + I_N) \qquad (5).$$

  Since I_i is a possible value of i–th intensity, we have $\max(0, \tilde{I}_i - \Delta_i) \leq I_i \leq \tilde{I}_i + \Delta_i$. So,

  $$I_N^+ - I_{k-1}^+ = (\tilde{I}_k + \Delta_k) + ...(\tilde{I}_N + \Delta_N) \geq$$
  $$I_k + ... + I_N, \qquad (6)$$

  and

  $$I_1 + ... + I_{k-1} \geq$$
  $$\max(0, \tilde{I}_1 - \Delta_1) + ... + \max(0, \tilde{I}_{k-1} - \Delta_{k-1}) =$$
  $$I_{k-1}^-. \qquad (7)$$

  From (5), (6), and (7), we conclude that

  $$I_{k-1}^- < q_0(I_N^+ - I_{k-1}^+).$$

  If we divide both sides of this inequality by

  $$I_N^+ - I_{k-1}^+,$$

  we conclude that $q_{k-1}^- < q_0$. Since r^- is by definition the smallest of the distances r_i for which $q_i^- \geq q_0$, we can conclude that $i > k-1$. Hence, $i \geq k$, and $r_k \leq r_i = r^-$, i.e., that $d = 2r_k \leq 2r^-$. The second statement is proven.

- To prove the third statement, let us take the value k for which $r^+ = r_k$, and take $I_i = \max(0, \tilde{I}_i - \Delta_i)$ for $i \leq k$ and $I_i = \tilde{I}_i + \Delta_i$ for $i > k$. Then, arguments like the ones used in the proof of the first statement show that for this k, the inequality (1) is equivalent to $q_k^+ \geq q_0$. Since by definition of r^+, k is the first integer for which this is true, we can thus conclude that the equivalent inequality (1) is also true. This, in its turn, means that the actual diameter d, defined as $2r_i$ for the first i for which (1) is true, is $\leq 2r_k = 2r^+$. So, in this case, $d \leq d^+$. On the other hand, we have already proven that in all cases (in particular, in this one), $d \geq d^+$. This means that in this case, $d = 2r^+$.
- The fourth statement is proved similarly, if we take as k, the value for which $r^- = r_k$, and take $I_i = \tilde{I}_i + \Delta_i$ for $i \leq k$, and $I_i = \max(0, \tilde{I}_i - \Delta_i)$ otherwise.

All four statements are proven. This completes the proof of our result.

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