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Invariance Leads to the Interval Character of Ordinal Statistical Characteristics

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Traditionally, in measurement theory, a unit of measurement is fixed, and the starting point determine the value of the quantity uniquely. However, in many real-life cases, we also have to describe quantities for which only the *order* makes sense: e.g., when a student is stronger than another, when a profession is more attractive than another, etc. In some of these cases, there may be a way to ascribe a meaningful numerical value to each degree of attractiveness, but, if such a method is not known, we have to be consent with the estimates for which only the order matters, i.e., which are defined modulo an arbitrary monotonic transformation.

In such cases, if we have several measurement results, we would like to describe some statistical characteristics. For these statistical characteristics to be meaningful, it is necessary to demand that they are invariant with respect to arbitrary monotonic transformations. This exclude such characteristics as, e.g., average, because average is not invariant: e.g., if in one scale x, we have three measurements 1, 2, and 3, then the average is 2. If we now use a different scale $y = 2^x$, the same measurements will be equal to 2, 4, and 8. If the average was invariant, then we would have the average in the new scale equal to $2^2 = 4$, but the actual average of these three values is $14/3 \neq 4$.

How to describe all possible numerical statistical characteristics for this case? By a *characteristic*, we mean a way to transform N given measurement results into a single value that is invariant with respect to arbitrary permutations (because the order of measurements is irrelevant), and arbitrary monotonic rescalings. The third reasonable demand is based on the fact that the measurements on which we base our characteristics may not be absolutely precise. Therefore, we base our computations of this characteristic on the *approximate* values of the measured quantity. It is therefore reasonable to demand that when we apply more and more precise measurements, the resulting value of the characteristic will be more and more accurate. In other words, it is reasonable to demand that the characteristic is expressed by a continuous function. So, we arrive at the following definition:

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Definition 1. *By an ordinal numerical statistical characteristic, we mean a continuous function $f : R^n \to R$ that satisfies the following two properties:*

- *It is invariant with respect to permutations, i.e., $f(x_{\pi(1)}, ..., x_{\pi(N)}) = f(x_1, ..., x_N)$ for an arbitrary permutation*
$$\pi : \{1, 2, ..., N\} \to \{1, ..., N\}.$$
- *It is invariant with respect to arbitrary monotonic rescalings ϕ, i.e., $f(\phi(x_1), ..., \phi(x_N)) = \phi(f(x_1, ..., x_N))$ for an arbitrary monotonically increasing 1–1 function $\phi : R \to R$.*

The description of all possible ordinal numerical statistical characteristics is basically, given in [2] (we give a slightly more general statement here):

PROPOSITION 1. *Every ordinal numerical statistical characteristic is of the type $x_{(k)}$ for some k.*

We have already mentioned that from finitely many experiments, we can only make an *approximate estimate* of the desired statistical characteristic. As a result of such an approximate estimation, we will have not a single numerical value, but the *interval* of possible values. How to describe the resulting *interval* characteristics?

Definition 2. *By an ordinal interval statistical characteristic, we mean a continuous function* **f** *from R^n into the set of all intervals that satisfies the following two properties:*

- *It is invariant with respect to permutations, i.e., $\mathbf{f}(x_{\pi(1)}, ..., x_{\pi(N)}) = \mathbf{f}(x_1, ..., x_N)$ for an arbitrary permutation*
$$\pi : \{1, 2, ..., N\} \to \{1, ..., N\}.$$
- *It is invariant with respect to arbitrary monotonic rescalings ϕ, i.e., $\mathbf{f}(\phi(x_1), ..., \phi(x_N)) = \phi(\mathbf{f}(x_1, ..., x_N))$ for an arbitrary monotonically increasing 1-1 function $\phi : R \to R$.*

PROPOSITION 2. *Every ordinal interval characteristic is of the type $[x_{(k)}, x_{(l)}]$ for some integers $k \le l$.*

Comment. These propositions describes all possible ordinal characteristics, i.e., characteristics that are defined modulo arbitrary monotonic rescalings. For these characteristics, intervals appear because we are trying to estimate the value of a statistical characteristic based on finitely many measurements. There is a special case when a numerical characteristics is simply impossible: the case when we try to estimate the *average* value of a sample (in some reasonable sense). A natural property of an "average" value is that that if we invert the order, e.g., change x_i to $-x_i$, then the average must also change its sign. So, we arrive at the following definition:

Definition 3.

- *An ordinal numerical characteristic*

$$f : R^N \to R$$

is called an average if for all x_i,

$$f(-x_1, ..., -x_N) = -f(x_1, ..., x_N).$$

- *An ordinal interval characteristic*

$$\mathbf{f} : R^N \to \mathbf{I}$$

is called an average if for all x_i,

$$\mathbf{f}(-x_1, ..., -x_N) = -\mathbf{f}(x_1, ..., x_N).$$

PROPOSITION 3.

- *If N is even, then there exists only one average ordinal numerical characteristic: the median $x_{(N/2)}$.*
- *If N is odd, then there exist no average ordinal numerical characteristics.*
- *Every average ordinal interval characteristic has the form $[x_{(k)}, x_{(N-k)}]$ for some $k \le N/2$.*

Comment. If N is even, then the narrowest possible estimate is a numerical estimate, namely, the median. What is the narrowest possible average for odd N? The answer to this question easily follows from the above proposition:

Definition 4. *An average ordinal interval characteristic $\mathbf{f}(x_1, ..., x_n)$ is called an (interval) median if it is the narrowest of all possible average ordinal interval characteristic $\mathbf{g}(x_1, ..., x_n)$ (i.e., if $\mathbf{f}(x_1, ..., x_n) \subseteq \mathbf{g}(x_1, ..., x_n)$ for all x_i).*

PROPOSITION 4. *For every N, there is exactly one interval median $\mathbf{f}$:*

- *When N is even, then*

$$\mathbf{f}(x_1, ..., x_n) = [x_{(N/2)}, x_{(N/2)}].$$

- *When N is odd, then*

$$\mathbf{f}(x_1, ..., x_n) = [x_{((N-1)/2)}, x_{((N+1)/2)}].$$

Comment 1. The lower and upper bounds of the interval median are called *left* and *right* medians.

Comment 2. For odd N, *there is no numerical characteristic that expresses the notion of the "average", only an interval one.*

A. Applications

This notion has been applied to the following areas:

A.1. Case study: sociology

Students from high schools of two different cities were asked to estimate their preference of different professions on a scale. The results from [5, 4] were analyzed first by using traditional statistical methods (average), which lead to a conclusion that the average preferences differed from city to city. However, since preferences are defined only modulo an arbitrary monotonic transformation, applying an arithmetic average makes no big sense. When interval medians were applied, the differences between high school students from different cities turned out to be insignificant [3].

A.2. Case study: quality measurements and quality control

In many cases, there is no objective measure of quality, so, we have to ask experts to estimate quality on a scale (from, say, 0 to 10). Then, if we want to compare the average quality assigned by experts, we have to use interval medians to describe these averages [1, 3].

B. Proofs

B.1. Proof of Proposition 1.

Indeed, let $x_i = i$. Let us choose the following function ϕ: $2x - 1$ for $x < 1$, $2x - N$ for $x > N$, $\phi(x) = i + 1.5(x - i)$ when $i \le x \le i + 0.5$, $i = 1, ..., N-1$, and $\phi(x) = i + 0.75 + 0.5(x - i - 0.5)$ when $i + 0.5 \le x \le i + 1$. This function is strictly increasing, 1–1, and $\phi(x) = x$ only for $x = 1, 2, ..., N$. Therefore, from the condition that $f(\phi(1), ..., \phi(N)) = \phi(f(1, ..., N))$, we conclude that $f(1, ..., N) = \phi(f(1, 2, ..., N))$ and therefore, that $f(1, ..., N) = k$ for an integer $k \in \{1, 2, ..., N\}$.

If the values x_i are all different, then, due to permutation invariance, we get $f(x_1, ..., x_N) =$

$f(x_{(1)},...,x_{(N)})$. One can easily construct a 1–1 monotonic transformation $\phi(x)$ that maps i into $x_{(i)}$ and therefore, $f(x_1,...,x_N) = f(x_{(1)},...,x_{(N)}) = f(\phi(1),...,\phi(N)) = \phi(f(1,...,N)) = \phi(k) = x_{(k)}$. So, for the case when all x_i are different, we get the desired equality. Cases when some of the values x_i coincide can be easily obtained from these ones by tending to a limit. So, since we assumed that f is continuous, this expression is true for all x_i. Q.E.D.

B.2. Proof of Proposition 2.

Indeed, it is easy to see that if $\mathbf{f} = [f^-, f^+]$ is an ordinal interval characteristic, then both f^- and f^+ are ordinal numerical characteristics, and thus, due to the previous Proposition, each of them can be described as $x_{(k)}$ for some k. Q.E.D.

B.3. Proof of Proposition 3.

This proposition follows from the fact that for $y_i = -x_i$, the ordering of $y_{(i)}$ is opposite to the ordering of $x_{(i)}$, i.e., $y_{(i)} = -x_{(N-i)}$. So, for interval characteristic $\mathbf{f}(x_1,...,x_n) = [x_{(k)}, x_{(l)}]$, the condition that $\mathbf{f}$ is an average means that $[x_{(k)}, x_{(l)}] = \mathbf{f}(x_1,...,x_n) = -\mathbf{f}(-x_1,...,-x_n) = -[y_{(k)}, y_{(l)}] = -[-x_{(N-k)}, -x_{(N-l)}] = [x_{(N-l)}, x_{(N-l)}]$. Therefore, $x_{(k)} = x_{(N-l)}$, and $l = N-k$. The statement about numerical characteristics follows if we take $k = l$. Q.E.D.

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