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On Application Of Interval Approach To Finding The Latent Singularity Points Of The Initial Value Problems

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I. The Problem of Finding The Singular Points

Let us consider the differential equation of first order

$$y' = f(x, y) \tag{1}$$

Suppose that the conditions of Picard-Lindeloef's theorem [1] are valid in some neighbourhood of the initial point (x_0, y_0).

Assume that the solution $y(x)$ that satisfies the initial condition $y(x_0) = y_0$ cannot be extended to all x. So, the maximal possible extension of this solution is a solution $y(x)$ that is defined on some (unknown) interval (a, x^*), where x^* is a *singularity point* for this solution. The problem is: how to compute x^*?

In this paper, we will consider the case when

$$y(x) \to +\infty \quad (x \to x^* - 0) \tag{2}$$

II. Orlov's Algorithm

For the case (2), the problem of computing the singular point x^* has been the subject of V. N. Orlov's research [2]. He constructed an auxiliary equation

$$u' = g(x, u) \tag{3}$$

by means of a transformation

$$u = 1/y. \tag{4}$$

Since $u' = -y'/y^2 = -f(x, y)/y^2$, we have $g(x, u) = -u^2 f(x, 1/u)$ (when $u \neq 0$). The equation (3) has the solution $u(x) = 1/y(x)$ on interval (b, x^*), where $y(x) > 0$, and b is some point between x_0 and x^*.

Assume that the function $g(x, u)$ is extended in such a way that $g(x, 0)$ exists, and the conditions of Picard-Lindeloef's theorem are valid in some rectangle containing the point $(x^*, 0)$. Then, $u(x)$ is well-defined for $x = x^*$, and $u(x^*) = 0$.

Thus, to compute x^*, it is sufficient to find zeros of corresponding solution $u(x)$ of the equation (3).

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This problem becomes easier if $u(x)$ changes its sign at the point x^*. The inequality

$$u'(x^*) = g(x^*, 0) \neq 0$$

is sufficient to guarantee this change of sign.

V. N. Orlov applied this method to Riccati equation (that are used, e.g., in *control theory*), and to first and second equation of Painlevé.

III. Orlov's Method Does Not Give Guarenteed Bounds For The Singularity Point

Orlov's method is approximate, and does not give a guaranteed estimates for x^*.

In this paper, we describe a two-sided modification of this method, a modification that uses interval approach and results in an interval that is *guaranteed* to contain x^*.

IV. Straightforward Interval Approach

At first glance, it may seem that it is sufficient to solve the "inverse" initial problem $u(x_0) = 1/y_0$, $u' = g(x, u)$ by one of interval methods. The integrating process passes through the values $x = x_0, x = x_1 = x_0 + \Delta x, x = x_2 = x_0 + 2 \cdot \Delta x, \ldots$ and gives the intervals $U(x_k)$, $k = 1, 2, \ldots$, which include $u(x_k)$. Suppose that

$$\begin{aligned} U(x_k) &> 0 \quad (k = 0, 1, \ldots, m), \\ 0 &\in U(x_k) \quad (k = m+1, \ldots, p), \\ U(x_{p+1}) &< 0 \end{aligned}$$

Then the interval (x_m, x_{p+1}) contains the zero of $u(x)$, i.e. the singular point x^* of $y(x)$.

V. Straightforward Interval Approach Does Not Always Work

The difficulties appear when $y(x)$ has zeros on the interval $[x_0, b)$. The described simple interval method is not suitable in this case.

VI. Application Of Co-Integration

In this case, a more reasonable idea is to use the integration process proposed in [3] and called "co-integration". In this process, we simultaneously solve both equations (1) and (3).

For $k = 0$, we start with $Y(x_0) = y_0$ and $U(x_0) = 1/y(x_0)$. After we have computed the intervals $Y(x_k)$ and $U(x_k)$ for some k, we do the following:

1) We run one step of two independent (and potentially parallel) interval processes for solving equations (1) and (3):
   - the first of these processes uses an interval $Y(x_k)$ (that is guaranteed to contain the unknown actual value $y(x_k)$) to compute the new interval $Y^0(x_{k+1})$ (that is guaranteed to contain $y(x_{k+1})$);
   - the second process uses an interval $U(x_k)$ (that is guaranteed to contain the unknown actual value $u(x_k)$) to compute the new interval $U^0(x_{k+1})$ (that is guaranteed to contain $u(x_{k+1})$);
2) Then, we apply the mutual correction by computing the following intersections:

$$Y(x_{k+1}) = Y^0(x_{k+1}) \cap (1/U^0(x_{k+1})),$$
$$U(x_{k+1}) = U^0(x_{k+1}) \cap (1/Y^0(x_{k+1})).$$

In this process, an unbounded interval $U(x_{k+1})$ can appear because of the location of zero near the point x_{k+1}, or because of the unperfection of underlying processes (for instance, a too large value of Δx). Therefore, in this method, we use a special interval software to deal with unbounded intervals.

The method does not have any problem with the possible zeros of the function $y(x)$. Near these zeros, the interval U has a tendency to be equal to $R = (-\infty, +\infty)$ and thus, to be excluded. The mutual correction makes the computed intervals narrower.

If simultaneously $Y(x_{k+1}) = U(x_{k+1}) = R$, then further integration has no sense. Then, we must decrease $|\Delta x|$ and renew the process either from the initial point x_0, or from some intermediate point x_k.

Decreasing Δx This is also a reasonable idea for the case when the resulting interval for x^* is too wide.

The transformation (4) is good for certain equations (1), but not for all of them. In general, each equation (1) has its own appropriate transformation $y \to u(y)$.

References

[1] E. Kamke, **Differentialgleichungen. Loesungsmethoden und Loesungen. 1. Gewoenlichen Differentialgleichungen**, Teubner, Leipzig, 1959.

[2] V. N. Orlov, **The investigation Of Approximate Solution With Mobile Poles Of Nonlinear Ordinary Differential Equations**, Ph.D. Dissertation, Minsk, 1989 (in Russian).

[3] G. G. Men'shikov, "Interval Co-Integration Of Differential Equations Connected By A Substitution Of The Variable In Interval Computation", 1992, No. 4(6), 1992.