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Validated Function Evaluation using Polynomial Approximation from Truncated Chebyshev Series

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The truncated Chebyshev series of a function can be used to get an efficient direct polynomial approximation for the function under consideration. Then the polynomial approximation can be evaluated by Horner's scheme.

To get a verified lower and upper bound for the exact function value, we must consider the following error components:

- the truncation error;
- the error of the coefficients of the truncated Chebyshev series;
- the error of the coefficients of the direct polynomial, as well as
- all rounding errors inherent in the floating-point evaluation of Horner's method.

A realistic upper bound for these errors (it can be derived a priori as we will demonstrate) can be used to formulate an interval routine which is able to compute an enclosure of the exact function range over an interval.

As an example, we will discuss a routine for the natural logarithm. The routine will be based on the rapidly converging Chebyshev series for the function

$$f(x) = \frac{x+1}{x-1} \cdot \ln(x).$$

Let $T_k(x)$ denote the k-th Chebyshev polynomial. If we denote

$$s = 3 - 2\sqrt{2}, \quad y = \frac{x-1}{s \cdot (x+1)} \quad \alpha = \arcsin(s),$$

then for all

$$x \in [\frac{1}{\sqrt{2}}, \sqrt{2}],$$

we have

$$f(x) = \sum_{k=0}^{\infty} c_{2k} T_{2k}(y),$$

where the coefficients c_{2k} can be generated by the following recurrent scheme:

$$c_0 := \frac{8\alpha}{s},$$

$$c_{2k+2} := \frac{8 \tan^{2k+1} \alpha}{(2k+1)s} - c_{2k}, \quad k = 0, 1, 2, \ldots$$

Using leading terms of this expansion, we compute the coefficients of this Chebyshev sum and the coefficients of the direct polynomial approximation by using multiple-precision interval arithmetic.

Finally, a priori error estimations considering all kinds of errors mentioned above will be shown, yielding an interval routine producing validated results on a computing device for all admissible (floating-point) interval arguments.

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