Fuzzy Interval-Valued Inference System with Para-Consistent and Grey Set Extensions

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Many-valued logic based interval reasoning plays increasingly important role in fuzzy and other many-valued extensions of traditional two-valued logic. It is often overlooked that many-valued logic reasoning has richer inference rule base than the classical crisp logics. Indeed some rules that are not possible in a point-based crisp logic (in which truth values are numbers) will have come to existence when we accept *intervals* as truth values, i.e., as the basic elements of the semantic space, with point values as special cases (that correspond to degenerate intervals).

Interval logic also naturally arises when to determine a fuzzy truth value $a = t(A)$ (from the interval $[0,1]$), we ask a community of n experts to vote, and take $a = u_1/n$, where u_1 is the total number of agents who voted in favor of A being true. This fuzzy value can be thus viewed as an *approximation* to the (crisp) $n-$dimensional logical vector that represents the actual results of voting. This interpretation of fuzzy values leads to the following interpretation of connectives: if we have a logical connective CON (e.g., AND, OR, etc.), then we can interpret $a\,CON\,b$ as follows:

- We find statements A and B for which the above procedure leads to truth values a and b, and
- We interpret the (similarly determined) truth value of $A\,CON\,B$ as $a\,CON\,b$.

The problem with this natural definition is that for one and the same pair of values (a,b), different choices of A and B are possible, and for these different choices, the truth values of $A\,CON\,B$ may be different. As a result, instead of a *single* truth value $a\,CON\,b$, this procedure leads to an *interval* of possible truth values. The lower and upper bounds of this interval will be denoted by BOT and TOP. So, for each connective from two-valued logic, we now have a pair of connectives from a multi-valued logic (e.g., a pair of AND connectives, implication operators etc.) whose values $CONBOT$ and $CONTOP$ form the natural bounds for $a\,CON\,b$ within $[0,1]$ valuation space. A formal semantics for such pairs that is derived by means of an exact mathematical method has been provided by the **checklist paradigm** [3, 4, 5, 6].

Interplay of formal techniques and conceptual issues has to be examined carefully, if one attempts to extend the interval-based inference to include essential *paraconsistent* features [9] (i.e., crudely speaking, the possibility for a statement and for its negation both have some degree of belief to be true). One of our aims is to analyze the epistemological and ontological status of some basic notions of checklist paradigm-based interval logic systems, so that we can analyze not only the valid inference, but also semantic conflicts in fuzzy logics. This requires also a careful foundational analysis of the systems involved. In such a foundational analysis we ought to distinguish sharply [8]

(1) mathematical questions,
(2) logical questions,
(3) ontological, epistemological and metaphysical questions,

when looking at the interrelationship of the pairs of connectives that form the interval semantic space.

In order to bring out the characteristic features of the metatheory of paraconsistency in approximate reasoning systems, we need to employ an adequate method of conceptual analysis. The purpose of this paper is to provide an up-to-date survey of some checklist paradigm techniques and extend the results in that direction.

In addition to the theoretical analysis presented here, some applications of fuzzy interval to knowledge representation will be discussed. The foundational concepts are needed as a basis for a sound methodology that could adequately trace not only the consistency of our requirements concerning the knowledge represented by formal structures, but also systematically examine the possible inherent contradictions in alternative designs of specific knowledge representation schemes. Once the contradictions and inconsistencies are discovered, one has to provide the adequate means for removing, or at least isolating and localizing, these.

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The interval inference that has been used in Activity Structures based Knowledge Processing Systems (such as knowledge based system CLINAID) has been theoretically justified by the checklist paradigm. The checklist paradigm generates *pairs of* distinct connectives of the same logical type that determine the end points of intervals, thus providing formally and epistemologically justified systems of *interval-valued* approximate inference. The most relevant further references are [3, 4, 6, 7, 12], each addressing a different aspect of the problem.

As said above (and presented in detail elsewhere), an interval fuzzy membership function can be derived within the checklist paradigm as an approximation of the crisp logical vector that represents a vote of a community of some formal agents. Given a crisp Boolean vector that represent this vote (a "fine structure"), a consensus summarization function for each vector and a measure $m(F)$ operating on a pair of such vectors, a "coarse structure" of many valued pair of interval connectives can be generated. This coarse structure is captured algebraically as an interval many-valued logic system, consisting of pairs of logical connectives of the same logical type that is used to manipulate the interval valuations of logical sentences.

We have shown elsewhere (Bandler and Kohout [3, 4, 5, 6]) that the *coarse* structure imposes bounds upon the fine structure, without determining it completely. Hence, *associated with the various logical connectives between propositions* **are** *their extreme values* that represent the interval valuations of logical sentences. Thus we obtain the inequality restricting the possible values of $m(F)$:

$$CONTOP \geq m(F) \geq CONBOT,$$

where CON is the name of connective represented by F. So, there are 16 such inequalities, as there are 16 **logical types** of CON, 10 of which are nontrivially two-argument.

Let us look now at some typical results. Choosing for the logical type of the connective CON the *implication* and making the assessment of the fuzzy value of the truth of a proposition by the formula (proposed, e.g., in [6])

$$m_1(F) = 1 - (u_{10}/n),$$

where u_{10} is the number of experts who voted "for" for the first statement A, and "against" the second statement B, we obtain:

$$\min(1, 1-a+b) \geq m_1(PLY) \geq \max(1-a, b).$$

We can see that for the $CONTOP$, the $PLYTOP$ was chosen, and the checklist paradigm produced the Lukasiewicz implication operator, and the other bound ($PLYBOT$) is the Kleene-Dienes implication operator.

Choosing for CON the connective type AND, we get:

$$\min(a, b) \geq m_1(AND) \geq \max(0, a+b-1).$$

Choosing for CON the connective type OR, we get:

$$\min(a+b-1, 1) \geq m_1(OR) \geq \max(a, b).$$

We have shown [4, 6] that for a given contraction/approximation measure $m_1(F)$, there are 16 inequalities linking the TOP and BOT types of connectives, as there are 16 logical types of connectives. Out of these 16 possible two-argument options, 10 are genuinely dependent on both arguments. The latter are of the following logical types:

- *conjunction* ($a\,OR\,b$),
- *disjunction* ($a\,AND\,b$),
- *non-conjunction* ($\neg(a\,OR\,b)$),
- *non-disjunction* ($\neg(a\,AND\,b)$),
- *implications* ($a \rightarrow b$, $a \leftarrow b$),
- *non-implications* ($\neg(a \leftarrow b)$, $\neg(a \rightarrow b)$),
- *equivalence* ($a \equiv b$),
- *non-equivalence* ($a \bigoplus b$), also known as *exclusive or* ($a\,EOR\,b$).

The minimal set of necessary conditions that the individual logical types of connectives have to satisfy are:

1. Duality of AND and OR.
2. A non-implication is obtained by negating an implication, and vice versa.

These conditions give general logical constraints on the systems of connectives. Taking a specific constraint measure, one obtains more specific results. For example, m_1 (defined as above) will yield 16 pairs $CONBOT \leq CONTOP$ specifying the end points of the interval for all the possible logical types of connective, some of which are listed below:

- For $\neg(a \rightarrow b)$, the valuations are
$$\max(b-a) \leq \min(1-a, b).$$
- For $\neg(a \leftarrow b)$, the valuations are
$$\max(a-b) \leq \min(a, 1-b).$$
- For $a \equiv b$, the valuations are
$$\max(a+b-1, 1-(a+b)) \leq \min(1-a+b, 1-b+a).$$
- For $a\,EOR\,b$, the valuations are
$$\max(a-b, b-a) \leq \min(a+b, 2-(a+b)).$$

- For $\neg(a\, OR\, b)$, the valuations are
$$\max(0, 1-a-b) \leq \min(1-a, 1-b).$$
- For $\neg(a\, AND\, b)$, the valuations are
$$\max(1-a, 1-b) \leq \min(1, 2-a-b).$$

For the exhaustive listing of all 16 connectives see, e.g., [4, 6].

The width of the interval produced by an application of a pair of associated connectives (i.e. TOP and BOT connectives) characterizes the margins of imprecision of an interval logic expression. The interval between the TOP connective and the bottom connective is directly linked to the concept of fuzziness ϕ. If we define the *unnormalized fuzziness of* x (cf. Bandler and Kohout 1978 [1]) as $\phi x = \min(x, 1-x)$, then for x in the range $[0,1]$, ϕx is in the range $[0,.5]$, with value 0 if and only if x is *crisp*, and value .5 iff x is .5. We have:

GAP THEOREM. (Bandler and Kohout, 1986, [6]):
$$a\, ANDTOP\, b - a\, ANDBOT\, b =$$
$$a\, ORTOP b - a\, ORBOT\, b =$$
$$a\, PLYTOP\, b - a\, PLYBOT\, b = \min(\phi a, \phi b).$$

$$a\, IFFTOP\, b - a\, IFFBOT\, b =$$
$$a\, EORTOP\, b - a\, EORBOT\, b = 2 \cdot \min(\phi a, \phi b).$$

Hence, the *margins of imprecision* can be directly measured by the degree of fuzziness ϕ.

Measures other than m_1 yield other interesting results as demonstrated by Bandler and Kohout in their 1980 paper [3]. If for F an implication operator type of connective is chosen again, but this time evaluation "by performance"
$$m_2 = u_{11}/(u_{10} + u_{11})$$
is used, this yields the inequality
$$\min(1, b/a) \geq m_2(F) \geq \max(0, (a+b-1)/a),$$
where $PLYTOP$ is in this instance the well-known G43 implication of Goguen-Gaines (cf., e.g., [3]).

Still another contracting measure
$$m_3 = u_{11} \bigvee (u_{00} + u_{01}),$$
where $\bigvee$ stands for max, yields [3]:
$$\max[\min(a,b), 1-a] \geq m_3(F) \geq \max(a+b-1, 1-a).$$

The lower "contrapositivization" of m_3 yields the measure
$$m_4 = (u_{11} \bigvee (u_{00} + u_{01})) \bigvee (u_{00} \bigvee (u_{01} + u_{11}))$$
that gives the following bounds [3]:
$$\min[\max(a+b-1, 1-a), \max(b, 1-a-b)] \leq m_4 \leq$$
$$\min[\max(1-a, b), \kappa a, \kappa b].$$
The measure $m_5 = m_2 \bigvee u_{00} + u_{11}$ yields:
$$\max[\min(1, b/a), 1-a] \geq m_5 \geq$$
$$\max[(a+b-1)/a, 1-a].$$

For the proofs of the results presented in this section and further explanation see [3], Sections 5 and 6.

Logic transformations are useful in investigating the mutual interrelationships of interval logic connectives. Let transformations on basic propositional functions $f(x,y)$ of 2 arguments be given as follows:
$$I(f) = f(x,y),\ D(f) = \neg f(\neg x, \neg y),$$
$$C(f) = f(\neg x, \neg y),\ N(f) = \neg f(x,y).$$

In the set of the above transformations $T_P = \{I, D, C, N\}$, the individual transformations are called *identity, dual, contradual, negation* transformation, respectively. It is known that for the crisp (2-valued) logic these transformations determine the Piaget group [13]. This group of transformations is a realization of abstract Klein 4-element group.

Adding new non-symmetrical transformations to those defined by Piaget enriches the algebraic structure of logical transformations [10, 11].

Adding
$$LC(f) = f(\neg x, y),\ RC(f) = f(x, \neg y),$$
$$LD(f) = \neg f(\neg x, y),\ RD(f) = \neg f(x, \neg y)$$
to the above defined four symmetrical transformations we obtain a new 8-element group of transformations
$$T = \{I, D, C, N, LC, RD, LC, RD\}.$$

The interval logic system based on m_1 can be characterized by such groups of transformations. If we are interested in extending the interval logics into the domain of paraconsistency, we have however base our systems on different measures, such as m_2 and such like.

One area where conflicts may arise is when we separate assertions and rejections of propositions. Assertion and rejection of a proposition belong properly to the domain of meta-operations. In those kinds of logic, for which the mutual duality of assertion and rejection are their meta-properties the object language defined ply operators will be *contrapositive*. The duality of inference rules (such as modus ponens and tollens) is assured by the contrapositive property of the ply operator [1, 3, 2] that enters the rules, i.e., by the property that for all a and b, the implications $a \to b$ and $\neg b \to nega$ have the same truth value: $a \to b = \neg b \to \neg a$.

One such system that is not contrapositive is the one that is determined by the measure m_2. The $\mathcal{S}_{2\times2\times2}$ group also brings some order to the connectives generated by this measure.

THEOREM. *The closed set of connectives generated from the ply-top implication operator* $a \to_4 b = \min(1, b/a)$ *by the transformation group* T *is listed below:*

$$\begin{aligned} g_1 &= I(\to_4) = \min(1, b/a), \\ g_2 &= C(\to_4) = \min(1, 1-b/1-a), \\ g_3 &= D(\to_4) = \max(0, b-a/1-a), \\ g_4 &= N(\to_4) = \max(0, a-b/a), \\ g_5 &= LC(\to_4) = \min(1, b/1-a), \\ g_6 &= LD(\to_4) = \max(0, 1-a-b/1-a), \\ g_7 &= RC(\to_4) = \min(1, 1-b/a), \\ g_8 &= RD(\to_4) = \max(0, a+b-1/a). \end{aligned}$$

This set of connectives together with T *is a realization of the abstract group* $\mathcal{S}_{2\times2\times2}$.

The corresponding implication operator operating on negative propositions, given by the formula $\max(0, (a+b-1)/a)$, also generates a similar group of transformations.

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