Applications of Interval Computations to Economic Input-Output Models

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***Abstract*— Economic input-output models are empirical realizations of general equilibrium economic models. They are particularly useful in determining how various industrial sectors of the economy are interrelated and in predicting how these sectors respond to changes in economic activity. In the United States, state and regional models are frequently used to help formulate local economic policy. Wassily Leontief is generally credited with the first useful implementation [2].**

I. Economic Input-Output Models

Let a_{ij} be the amount of the i–th good used to produce one unit of the j–th good. Let x_j be the amount of the j–th good produced, and b_j be the amount of final (consumer) demand for the j–th good. Demand for the i–th good is $\sum a_{ij}x_j + b_i$, and in equilibrium (supply = demand)

$$x_i = \sum_{i=1}^{n} a_{ij}x_j + b_i.$$

Equilibrium for all goods in matrix form is

$$x = Ax + b,$$

so

$$(I - A)x = Lx = b,$$

where L is called the *Leontief matrix*. If final demand is known, then the amount of the goods needed to satisfy this demand can be found. In practice these models are implemented using the dollar value of output of industrial sectors rather than individual goods - there are just too many goods. The computer industry is an example of an industrial sector. Some of the output from this industry will be used as inputs used to produce other goods and some will be sold as final goods (b).

Rohn [6] has derived the conditions under which the input-output system will have economically feasible solutions. For our purposes, it is sufficient to note that the Leontief matrix is most likely an M-matrix. Indeed, all of the production coefficients a_{ij} must be non-negative (a negative value would mean that a negative amount of good i is used to produce j). Hence, the off diagonal elements of the Leontief matrix will be negative.

Further, we must have $Lu > 0$ for some positive vector u. For our case, we can take $u = (1, 1, \ldots, 1)'$. Indeed, consider the product of the first row of the Leontief matrix with u (these arguments will hold for all rows). This gives

$$1 - a_{11} - a_{12} - \cdots - a_{1n}. \qquad (1)$$

The a_{1j} terms represent the proportions of good 1 used as inputs in industrial processes. A negative result for (1) would mean that more of good 1 is being used than is being produced. A zero result for (1) might be possible. This would mean that a good has no consumer demand but is used only as an input in production processes. Recall that the practical implementation is in industrial rather than goods terms. In practice this will mean that a zero result will not occur. Even if it did, there is an easy fix: aggregate this industry will one closely aligned that has some consumer demand.

Because L is an M-matrix, the inverse and hull solutions of the input-output model exist [4] and

$$L^{-1}b = L^H b = [\overline{L}^{-1}\underline{b}, \underline{L}^{-1}\overline{b}] \qquad \text{if } b \geq 0$$

The condition $b \geq 0$ is met because consumer demand must be non-negative.

The inverse Leontief matrix offers some efficiencies. It is useful to avoid solving $Lx = b$ for many different b. The solution requires that final demand, b, be forecast - a formidable task in itself. Much knowledge can be gained by considering how economic activity will change if there is a change in final demand. This does not require computing the level of economic activity, only the change. One method of examining how a change in final demand will affect an economy is to construct so-called *multipliers*.

As an example of an elementary multiplier con-

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sider the solution x of an equation

$$Lx = (0, 1, \ldots, 0)',$$

which shows the effect on the economy if there is a unit increase in final demand for good 2. But this solution is equal to

$$\sum_{i=1}^{n} l^{i2}$$

where l^{ij} is an element of the Leontief inverse L^{-1}. By considering the multiplier, we do not have to solve the system n times.

The simple output multiplier discussed above understates the amount of activity that will occur for a unit change in final demand [3]. An initial increase in final demand will increase output. This increase in output will increase household income which will further increase final demand. The resulting multiplier is called a Type II multiplier.

The Type II multiplier is computed by forming a new A matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & a_{1h} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{h1} & a_{h2} & \cdots & a_{hn} & a_{hh} \end{pmatrix}$$

where the h index is used to indicate households. The elements a_{ih} represents household purchases of output from the i–th industry. The elements a_{hi} represent payments by the i–th industry to households (wages, rent, interest, profits). The change in income generated by the i–th industry is

$$\frac{l^{hi}}{l_{hi}}$$

where l_{hi} and l^{hi} are elements of the Leontief and Leontief inverse matrices, respectively.

II. Why Interval Computations may be Useful

The technical coefficients l_{ij} of the Leontief matrix are not known precisely. The U.S. Department of Commerce conducts a census of U.S. businesses. These data are used by the Bureau of Economic Analysis to construct these coefficients for the U.S. economy. Even if the data from the census is not subject to error, it takes about 5 years to produce a new table. The Department of Commerce uses these tables to estimate Gross National Product (GNP). The values of the technical coefficients will change during this time period.

When the Department shifted from the 1972 census benchmark to the 1977 benchmark this raised the 1983 GNP estimate by $100 billion. Errors of this magnitude can lead policy makers to inappropriate action (e.g., creating inflation by fighting a non-existent recession).

Things are even worse for regional analysts who rarely have the resources to even conduct surveys of regional businesses. Usually they are forced to infer the regional technical coefficients by adjusting national tables and using other ad hoc procedures [5]. I think it is reasonable to believe that the technical coefficients are subject to some level of uncertainty.

An additional problem facing input-output analysts is that the final demand, b, is also not known with precision but must be forecast. This creates additional uncertainty in the Leontief system. In practice, I believe the effects of this uncertainty is largely ignored (although the pioneer work [6] that proposed the use of intervals for Leontief models appeared as early as 1980). Interval analysis can offer an approach to dealing with these problems. early as 1980, interval methods are still not used by economists). Users will probably have to subjectively assign levels of uncertainty using some ad hoc procedure. One possibility is to let all the elements of the Leontief matrix vary by some percentage.

III. An Example

Miernyk [3] uses a small, hypothetical input-output model with six producing sectors and a household sector to illustrate some of the mechanics of input-output analysis. We can use this hypothetical model as a partial check on the interval results for the Type II multiplier. Note that, in practice, the input-output matrix produced by IMPLAN (a commercial package) for Coconino County, Arizona contains 178 producing sectors which would be far to large to reproduce here.

The A matrix presented below is that given by Miernyk. Arbitrarily, I created three interval matrices symmetrical about Miernyk's matrix and computed the interval Type II multipliers. The resulting interval multipliers are shown below. The results were computed using C-XSC. I expected the interval multiplier to enclose Miernyk's point multipliers.

The interval matrix with a small level of uncertainty ($A \pm 0.001A$) did **not** enclose Miernyk's results. This caused some concern. Further investigation using C-XSC and two other (point) matrix inversion programs showed that Miernyk's Leontief inverse was wrong. The following interval matrices do enclose the results that Miernyk should have got.

$$A = \begin{pmatrix} 0.16 & 0.26 & 0.03 & 0.05 & 0.13 & 0.13 & 0.19 \\ 0.08 & 0.07 & 0.18 & 0.03 & 0.08 & 0.18 & 0.24 \\ 0.11 & 0.04 & 0.21 & 0.03 & 0.13 & 0.07 & 0.07 \\ 0.17 & 0.02 & 0.05 & 0.21 & 0.16 & 0.09 & 0.06 \\ 0.06 & 0.00 & 0.03 & 0.36 & 0.08 & 0.04 & 0.12 \\ 0.03 & 0.11 & 0.18 & 0.15 & 0.05 & 0.13 & 0.11 \\ 0.25 & 0.32 & 0.18 & 0.13 & 0.18 & 0.20 & 0.01 \end{pmatrix}$$

$A \pm 0.001A$

Industry	Multiplier	Diameter
1	$[4.81, 4.88]$	0.07
2	$[3.79, 3.85]$	0.05
3	$[6.34, 6.44]$	0.09
4	$[9.20, 9.35]$	0.15
5	$[6.04, 6.13]$	0.09
6	$[5.71, 5.80]$	0.08

$A \pm 0.01A$

Industry	Multiplier	Diameter
1	$[4.51, 5.22]$	0.71
2	$[3.57, 4.10]$	0.52
3	$[5.94, 6.89]$	0.95
4	$[8.58, 10.05]$	1.47
5	$[5.65, 6.56]$	0.91
6	$[5.35, 6.20]$	0.85

$A \pm 0.1A$

Industry	Multiplier	Diameter
1	$[2.62, 12.84]$	10.21
2	$[2.15, 9.69]$	7.54
3	$[3.39, 17.18]$	13.79
4	$[4.71, 26.17]$	21.46
5	$[3.21, 16.49]$	13.28
6	$[3.09, 15.34]$	12.26

IV. Other Approaches

West [7] has derived a formal expression for the probability density of the multipliers assuming that the technical coefficients are independent and that they can be characterized by probability distributions with small variances. This is certainly a significant improvement over assuming the coefficients are point values and that the multiplier results are known with certainty. If probability distributions are assumed, then West's method will generate confidence intervals.

Yet significant practical problems remain. We do not know what the probability distributions of the coefficients are in practice. Since the national tables are at least 5 years old before publishing, it is reasonable to believe that the distribution has changed. The regional analyst usually has no sample information at all. The regional analyst, most likely, will have no basis for assigning confidence intervals, but will have to assign subjective levels of uncertainty. My own feeling is that national analysts should be most suspicious about the possibility of assigning confidence intervals, given the delay in obtaining data.

V. Associated Research

Recent economic research has shown that most of the elements of the Leontief matrix are not "important". That is, they could be given zero value and this would have little affect on the resulting multipliers [1]. Extensions of this research will be useful because it will identify those elements in the Leontief matrix where resources should be directed to try to determine the level of uncertainty in the parameters.

References

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