# Quality Improvement Via the Optimization of Tolerance Intervals During the Design Stage

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In a world where competition for markets is ever more intense, ensuring a satisfactory quality to products at the lowest possible cost is a vital issue. This is the main reason why industry finds statistical quality control increasingly appealing. Until recently, most of the effort has been concentrated on the control of the quality of products at the end of the production line. It is one of the merits of G. Taguchi's celebrated work (see e.g. [1, 2]) to have stressed that much money could be saved by taking quality requirements into account right from the design stage. By combining elementary statistical notions with a good engineering intuition, he showed how to make products robust to fluctuations in the quality of their components. Good quality products can then be obtained by combining low quality (i.e. cheap) components. Some criticisms have been expressed about several aspects of Taguchi's approach. They include the following:

- the criterion used is not clearly related to the problem to be solved,
- too many experiments are required,
- there is no explicit optimization, due to the absence of a mathematical model of the process,
- interactions between factors are difficult to handle.

Combining Taguchi's ideas with the well-established response surface methodology, the model-based approach to quality improvement during the design stage developed by I. Vuchkov and L. Boyadjieva (see e.g. [3, 4]) can be seen as a major improvement to Taguchi's approach on these four counts. The resulting method involves the construction of an explicit model $\boldsymbol{\eta}(\boldsymbol{f}, \boldsymbol{\theta})$, linear in its parameters $\boldsymbol{\theta}$, where $\boldsymbol{\eta}(.,.)$ defines the performance characteristic(s) of interest, $\boldsymbol{f}$ is a vector of factors, and where the parameter vector $\boldsymbol{\theta}$ may be given *a priori* or estimated from experiments performed in controlled conditions, i.e. with $\boldsymbol{f}$ known exactly. In mass production, the factors cannot be set precisely to their desired levels but fluctuate randomly, e.g. due to the variability of the product components. This is why $\boldsymbol{f}$ is then assumed to be distributed according to the normal distribution $\mathcal{N}(\boldsymbol{x}, \Sigma)$, with the covariance $\Sigma$ equal to some known (e.g. diagonal) matrix and with the averages $\boldsymbol{x}$ (the nominal value of the factors) to be optimized. The model $\boldsymbol{\eta}(., \boldsymbol{\theta})$ is then used to construct predictors of the mean and variance of the performance characteristics. These predictions permit for instance to find the feasible nominal value for the factors that minimizes an approximation of the mean square error

$$MSE(\boldsymbol{x}) = E_{\boldsymbol{f}} \|\boldsymbol{y}^* - \boldsymbol{\eta}(\boldsymbol{f}(\boldsymbol{x}), \boldsymbol{\theta})\|_2^2 ,$$

where $\boldsymbol{y}^*$ is the desired value of the performance characteristics.

The aim of this paper is to present a deterministic counterpart to this model-based statistical approach. It will permit to take into account fluctuations that are characterized in terms of tolerance intervals without having to approximate them as gaussian random variables. It must be noted that information on the fluctuations of the characteristics of components is most often expressed in terms of tolerance intervals, so that an interval approach to quality improvement seems highly desirable (for instance, tolerance intervals on resistors are specified in terms of percentages of nominal values). Moreover, whenever components are sorted according to their tolerances from some initial population, the factors corresponding to the components with the lowest quality (i.e. the largest tolerance intervals and the smallest price) will be far from following a normal distribution, even if the initial population was approximately gaussian.

The paper will be organized as follows. Attention will be restricted to the scalar case, with $\eta(\boldsymbol{f}, \boldsymbol{\theta})$ linear in $\boldsymbol{\theta}$ and quadratic in $\boldsymbol{f}$ (for the problem to have any meaning, it is essential that $\eta(.,.)$ be nonlinear in the factors, so that nonlinearity can be put at work to compensate for their fluctuations). In Section 2, the model $\eta(., \boldsymbol{\theta})$ is assumed to be known *a priori*, and no uncertainty on $\boldsymbol{\theta}$ is taken into account. This model may for instance have been con-

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structed according to the response surface methodology. The criterion to be minimized is then

$$j(\boldsymbol{x}) = \max_{\boldsymbol{f} \in \mathcal{F}(\boldsymbol{x})} |y^* - \eta(\boldsymbol{f}, \boldsymbol{\theta})| ,$$

where $\mathcal{F}(\boldsymbol{x})$ is the vector interval (i.e., the axis-aligned parallelepiped, also known as a *box*) defining the tolerances in mass production around the nominal value $\boldsymbol{x}$ of the factors. Section 3 considers the case where $\boldsymbol{\theta}$ is estimated from measurements obtained in controlled conditions, where the $i$th observation satisfies

$$y_i = \eta(\boldsymbol{x}_i, \boldsymbol{\theta}) + \epsilon_i \; , \quad i = 1, \ldots, N \; ,$$

with $\epsilon_i$ belonging to some known uncertainty interval and $\boldsymbol{x}_i$ known without error. A parameter bounding methodology will be used to build the set $\mathcal{S}^N$ of all values of $\boldsymbol{\theta}$ that are consistent with these experimental data. The criterion to be minimized then becomes

$$j(\boldsymbol{x}) = \max_{\boldsymbol{\theta} \in \mathcal{S}^N, \boldsymbol{f} \in \mathcal{F}(\boldsymbol{x})} |y^* - \eta(\boldsymbol{f}, \boldsymbol{\theta})| \, .$$

It takes into account, in a guaranteed way, the fluctuations of the quality of the product components as well as the uncertainty on the model parameters. Section 4 addresses the situation where measurements are performed with errors on the factors, i.e.

$$y_i = \eta(\boldsymbol{f}_i, \boldsymbol{\theta}) + \epsilon_i \; , \quad i = 1, \ldots, N \; ,$$

with $\epsilon_i$ as in Section 3 and $\boldsymbol{f}_i = \boldsymbol{x}_i + \boldsymbol{\phi}_i$, where $\boldsymbol{\phi}_i$ belongs to some vector uncertainty interval. It will be shown that when $\eta(\boldsymbol{f}, \boldsymbol{\theta})$ is quadratic in $\boldsymbol{f}$, the set $\mathcal{S}^N$ can be included in a finite union of convex polyhedra (of polytopes if trivial to check identifiability conditions are met). This permits the replacement of the set $\mathcal{S}^N$ by an approximation guaranteed to contain it. A very simple example will illustrate these notions in Section 5. Remaining open questions will be offered for discussion.

## References


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