Convex Modeling - A Generalization Of Interval Analysis For Nonprobabilistic Treatment Of Uncertainty

Isaac Elishakoff

***Abstract*— This paper reviews some new developments is convex modeling of uncertainty in structural approaches to engineering. Convex modeling, on one hand, represents an alternative and complement to traditional probabilistic methods, and on the other hand, it is a generalization of the mathematical theory of interval analysis.**

I. Probabilistic Models

A structure can be modeled as a system subject to influences, disturbances, excitations, and so forth, designated as *inputs.* We are usually interested in the response of the system: its dynamic displacements, stresses, or strains designated as *outputs.* In formulating the probabilistic model, we first identify the random variables and/or functions, which can be the characteristics of the inputs and/or of the system itself. To perform the analysis, we need to know the joint probability density function of these variables. Very often the experimental information on the probability densities is lacking, and the researchers are making unsubstantiated and unverified assumptions on the density and distribution functions. The probabilistic analyst in actuality claims: "Give me the joint probability densities of random variables involved, and I will calculate the probability of the mechanical system performing its designated mission with desired accuracy!" This reminds us of the well-known statement by Archimedes: "Give me a firm spot on which to stand, and I will move the Earth". It should be borne in mind that, as Wentzel [36] notes, probabilistic models are "...viewed to be a sort of magic wand yielding information from nothing, i.e., from total ignorance. Those who think so are under a misapprehension since probability theory is used but to transform data on observed phenomena to infer the behavior of those which cannot be observed."

The author is with the Center for Applied Stochastics Research, Department of Mechanical Engineering, Florida Atlantic University, Boca Raton, FL 33431-0991. This study has been supported by the National Science Foundation under award No. MSS-9215698 (Dr. R. C. Chong – Program Director). Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the author and do not necessarily reflect the views of the National Science Foundation.

II. Convex Modeling

Very often, the information on uncertain variables is fragmentary. Indeterminacy about the uncertain variables involved could be formulated in terms of these variables belonging to certain sets, such as:

a) The uncertain parameter x_j is bounded:
$$\underline{a}_j \leq x_j \leq \overline{a}_j,$$
where $\underline{a}_j$ and $\overline{a}_j$ are lower and upper values, respectively, that the variable x_j can take.

b) The uncertain function has envelope bounds:
$$\underline{x}(t) \leq x(t) \leq \overline{x}(t),$$
where $\underline{x}(t)$ and $\overline{x}(t)$ are deterministic functions which delimit the range of variation of $x(t)$.

c) The uncertain function has an integral square bound:
$$\int x^2(t)\,dt \leq a.$$

d) The Fourier coefficients A of the uncertain function $x(t)$ belong to an ellipsoidal set
$$A^T W A \leq a^2,$$
where W is the weighing matrix.

All above models represent convex sets. As is seen, the information contents of a convex model is usually a bound or constraint which defines an infinite set of values of functional forms which an uncertain variable may assume. Let us consider a set-theoretic approach to modeling the uncertainty of a time dependent macroscopic vector function f. Let Γ be a set of vector-values functions. For a positive integer n, consider the set of functions:

$$F_n = \{f \mid f(t) = \frac{1}{n}\sum_{i=1}^{n} g_i(t) \text{ for } g_i \in \Gamma, i = 1, 2, ..., n\}.$$

F_n is the set of $n-$fold averages of vector functions in Γ. It is known [1] that as $n \to \infty$, the sequence of sets F_n converges to the convex hull of Γ. Or, in physical terms, if a macroscopic time-dependent vector $f(t)$ is formed as the superposition of numerous microscopic time-varying events $g_i(t)$ chosen from a set Γ, then the set of all such functions $f(t)$ will tend to be convex, regardless of the structure of the set Γ.

The convex modeling of uncertainty produces:

- maximum, or least favorable response of the structure, and
- minimum, a best favorable response.

It was applied to:

- vehicles moving on the rough surface with bounded terrain [12, 15];
- stress concentration in elastic sheets with bounded irregularities [29];
- optimization of trusses [23],
- and several other problems.

III. Convex Modeling Represents a Generalization of Interval Analysis

Indeed, type a) of the above four constraints is a representation of an interval variable. For this type of convex models, one can utilize the ample knowledge accumulated in interval analysis:

- Elishakoff and Duan [18] have utilized interval analysis to deal with interval masses and stiffnesses of the system.
- Analogously, Qiu et al [33, 34] developed interval results for natural frequencies, whereas
- Köylüoglu et al [31] developed interval finite element analysis.

The lecture presents some recent developments of convex (and, in particular, of interval) analysis of structures to analyze uncertainty, and future research needs.

References

[1] Z. Artstein, "On the calculus of closed set-valued functions", *Indiana University Mathematics Journal*, 1974, Vol. 24, pp. 433–441.

[2] Y. Ben-Haim and I. Elishakoff, "Dynamics and failure of a thin bar with unknown but bounded imperfections", In: D. Hui and N. Jones (eds.), **Recent advances in impact dynamics of engineering structures**, 1989, AND-Vol. 105, AD-Vol. 17, ASME, N.Y., 1989, pp. 89–96.

[3] Y. Ben-Haim and I. Elishakoff, "Non-probabilistic models of uncertainty in the buckling of shells with general imperfections: theoretical derivation of the knockdown factor", *Journal of Applied Mechanics*, 1989, Vol. 56, pp. 403–410.

[4] Y. Ben-Haim and I. Elishakoff, **Convex models of uncertainty in applied mechanics**, Elsevier Publ., Amsterdam, 1990.

[5] Y. Ben-Haim and I. Elishakoff, "Convex models of vehicle response to unknown but bounded terrain", In: H, Chung (ed.), *Proc. of the ASME Pressure Vessels and Piping Conference*, Honolulu, Hawaii, 1989, pp. 81–88; extended version: *Journal of Applied Mechanics*, 1991, Vol. 58, pp. 354–361.

[6] F. L. Chernousko, **State Estimation for Dynamic Systems**, CRC Press, Boca Raton, 1994.

[7] I. Elishakoff, "Convex versus probabilistic modeling of uncertainty in structural dynamics", In: M. Petyt, H. F. Wolfe, and C. Mei (eds.), **Structural Dynamics: Recent Advances**, Elsevier Applied Science Piblishers, Lonodn, 1991, pp. 3–21.

[8] I. Elishakoff, "Some questions in engineering eigenvalue problems in natural sciences", In: J. Albrecht, L. Collatz, P. Hagedorn, and W. Velte (eds.), **Numerical Treatment of Eigenvalue Problems**, Intl. Series of Numerical Mathematics, Vol. 96, Birkhäuser Publishers, Basel, Switzerland, 1991, pp. 71–107.

[9] I. Elishakoff, "Essay on probability, convexity, and earthquake engineering", *Proceedings of the 5th Italian National Conference on Earthquake Engineering "L'Ingegneria Sismica in Italia"*, 1991, Vol. 2, pp. 907–935.

[10] I. Elishakoff, "Essay on reliability index, probabilistics interpretation of safety factor, and convex models of uncertainty", In: F. Casciati and J. B. Roberts (eds.), **Reliability Problems: General Principles and Applications in Mechanics of Solids and Structures**, Springer Verlag, 1991, pp. 237–271.

[11] I. Elishakoff, "Essay on uncertainties in elastic and viscoelastoc structures: from A. M. Freudental's criticisms to modern convex modeling", *Intl. Journal of Computers and Structures*, 1995 (to appear).

[12] I. Elishakoff and Y. Ben-Haim, "Probabilistic and convex models of uncertainty in structural dynamics", *Proceedings of the 8th Intl. Modal Analysis Conference*, Orlando, 1990, pp. 1487–1492.

[13] I. Elishakoff and Y. Ben-Haim, "Dynaimcs of a thin cylindrical shell under impact with limited deterministic information on its initial imperfections", *Journal of Structured Safety*, 1990, Vol. 8, pp. 103–112.

[14] I. Elishakoff, G. Q. Cai, and J. H. Starnes, Jr., "Non-linear buckling of a column with

initial imperfections via stochastic and non-stochastic, convex models", *Intl. Journal of Non-Linear Meahcnics*, 1994, Vol. 29, pp. 71–82.

[15] I. Elishakoff, G. Q. Cai, and J. H. Starnes, Jr., "Probabilistic and convex models of uncertainty in buckling of structures", In: G. I. Schuëller, M. Shinozuka, and J. Yao (eds.), *Proceedings of the 6th Intl. Conference on Structural Safety and Reliability*, Balkema Publ., Rotterdam, 1994, Vol. 2, pp. 761–766.

[16] I. Elishakoff and P. Colombi, "Combination of probabilistic and convex models of uncertainty when scarce knowledge is present on acoustic excitation parameters", *Computer Methods in Applied Mathematics and Engineering*, 1993, Vol. 104, pp. 187–209.

[17] I. Elishakoff and P. Colombi, "Ideas of probability and convexity combined for analyzing parameter uncertainty", In: G. I. Schuëller, M. Shinozuka, and J. Yao (eds.), *Proceedings of the 6th Intl. Conference on Structural Safety and Reliability*, Balkema Publ., Rotterdam, 1994, Vol. 2, pp. 109–113.

[18] I. Elishakoff and Dehe Duan, "Application of the mathematical theory of interval analysis for uncertain vibrations", In: J. M. Cuschieri, S. A. L. Glegg, and D. M. Yeager (eds.), *Proceedings of the 1994 National Conference on Noise Control Engineering*, N.Y., 1994, pp. 519–524.

[19] I. Elishakoff, P. Eliseeff, amd S. Glegg, "Convex modeling of material uncertainty in vibrations of a viscoelastic structure", *American Institute of Aeronautics and Astronautics (AIAA) Journal*, 1994, Vol. 32, pp. 843–849.

[20] I. Elishakoff and J. J. Fang, "Diagnosis of local modifications in the boundary conditions of a rectangular plate", *Computer Methods in Applied Mechanics and Engineering*, 1995 (to appear).

[21] I. Elishakoff and J. J. Fang, "A new safety factor based on convex modeling", *Journal of Structural Safety*, 1995 (to appear).

[22] I. Elishakoff, Yosi Gana-Snvili, and D. Givoli, "Treatment of uncertain imperfections as a convex optimization problem", In: L. Esteva and S. E. Ruiz (eds.), *Proceedings of the Sixth Intl. Conference on Applications of Statistics and Probability in Civil Engineering*, Mexico, 1991, Vol. 1, pp. 150–157.

[23] I. Elishakoff, R. T. Haftka, and J. J. Fang, "Structural design under bounded uncertainty optimization with antioptimization", *Intl. Journal of Computers and Structures*, 1995 (to appear).

[24] I. Elishakoff, C. Q. Li, and M. Shinozuka, "Hybrid probabilistic and convex modeling for robust reliability analysis of structures", *Intl. Journal of Solids and Structures*, 1995 (to appear).

[25] I. Elishakoff, Y. W. Li, and J. H. Starnes, Jr., "A deterministic method to predict the buckling locad variability due to uncertain elastic moduli", *Computer methods in Applied Mechanics and Engineering*, 1994, Vol. 111, pp. 155–167.

[26] I. Elishakoff, Y. K. Lin, and L. P. Zhu, **Probabilistics and convex modeling of acoustucally excited structures**, Elsevier Science, Amsterdam, 1994.

[27] I. Elishakoff and B. Pletner, "Analysis of base excitation as an uncertain factor with specific bounds on it and its derivatives", In: T. C. Huang et al (eds.), **Structural Vibrations and Acoustics**, ASME Press, 1991, pp. 177-184.

[28] J. J. Fang and I. Elishakoff, "Convex identification of boundary conditions by the finite element method", *Journal of Shock and Vibration*, 1995 (to appear).

[29] D. Givoli and I. Elishakoff, "Stress concentration at a nearly circular hole with uncertain irregularities", ASME Paper 91–WA/APM–22, *Journal of Applied Mechanics*, 1992, Vol. 59, pp. 65–71.

[30] D. Givoli, I. Elishakoff, and Y. Stavsky, "A boundary-perturbation finite-element method for plane elasticity problems", *Computer methods in Applied Mehcanics and Engineering*, 1992, Vol. 96, pp. 45–63.

[31] H. U. Köylüoglu, A. S. Cakmak, and S. R. K. Nielsen, **Applications of interval mapping in mechanics for structural uncertainties and pattern loading**, Department of Building Technology and Structural Enginering, Aaborg University, Ppare No. 121, 1994.

[32] Y. W. Li, I. Elishakoff, J. H. Starnes, Jr., and M. Shinozuka, "Prediction of natural frequency and buckling load variability by convex modeling", *Fields Institute Communications*, American Math. Society, 1995 (to appear).

[33] Z. P. Qiu, S. H. Chen, and J. X. Na, "The Rayleigh quotient method for computinfg eigenvalue bounds of vibrationsl systems with interval parameters", *Acta Mehcanica Sinica*, 1993, Vol. 6, pp. 309–318.

[34] Z. P. Qiu, S. H. Chen, and I. Elishakoff, "Natural frequenices of structures with uncertain but

nonrandom parameters", *Journal of Optimization Theory and Applications*, 1995 (to appear).

[35] F. C. Schweppe, **Uncertain Dynamical Systems**, Prentice Hall, Englewood Cliffs, NJ, 1973.

[36] E. Wentzel, **Operations Research**, Mir Publishers, Moscow, 1988, p. 28.