A Posteriori Error Estimation and Corrected Solution of Partial Differential Equation

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***Abstract*— In this paper, we show that a posteriori error estimate technique can be used to produce corrected solutions of boundary value problem for partial differential equations.**

I. Formulation Of The Boundary Value Problem

We will illustrate our idea on the following elliptic boundary value problem

$$Lu = f(x,u), \quad x \in \Omega, \tag{1}$$
$$u(x) = 0, \quad x \in \partial\overline{\Omega}, \tag{2}$$

where Ω is a bounded open convex domain in R^2, with piecewise smooth boundary $\partial\overline{\Omega}$, and

$$Lu = \sum_{i=1}^{2} \partial^2 u / \partial x_i^2.$$

In this paper, we assume that

$$\partial f(x,u)/\partial u \;\geq\; q(x) \geq 0,$$

and that a positive constant K exists such that

$$|f(x,\eta)| \leq K(1+|\eta|),$$

for all $x \in \Omega$, and for all $\eta \in [\min_\Omega u, \max_\Omega u]$.

The solution of (1) is understood in the following weak sense: find a function $u \in W_2^1(\Omega)$ such that

$$L(u,v) = (f,v), \quad \forall v \in W_2^1(\Omega), \tag{3}$$

where $(\cdot,\cdot)$ is the inner product in L_2:

$$L(u,v) = \int_\Omega \sum_{i=1}^{2} \partial_i u \, \partial_i v \, \mathrm{d}\Omega.$$

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II. A Numerical Solution Of The Boundary Value Problem

To describe the solution, we must first define the finite element space S_l^n. We fix a mesh size h, and divide the domain Ω into subdomains of size h. The set of these subdomains will be denoted by $\mathcal{T}_h$. The finite-element space is defined as a set of piece-wise polynomial functions (i.e., function that are polynomial on each of the subdomains $T \in \mathcal{T}_h$):

$$S_l^n = \{\, s(x) \mid s \in W_2^l(\overline{\Omega}), s|_T \in \mathcal{P}^n, T \in \mathcal{T}_h \}, \tag{4}$$

where $\mathcal{P}^n$ is the set of polynomials of degree n.

We define the finite element solution u^h of the problem (1) as a function from S_1^1 that satisfies the conditions

$$L(u^h, v^h) = (f(x, u^h(x)), v^h), \forall v^h \text{ in } S_1^1, \tag{5}$$
$$u_h = 0, \quad \text{on } \partial\overline{\Omega}.$$

Since the function u^h is uniquely determined by its (finitely many) polynomial coefficients, we have a (non-linear) system of equations, that can be solved, e.g., if we use iterative techniques.

III. Estimating Error Of This Numerical Solution

We would like to use the *defect*

$$Lu^h - f(x, u^h)$$

of the approximate solution u^h to estimate its accuracy, i.e., the difference between this approximate solution and the (unknown) precise one. However, we cannot compute the defect of the original solution, because by definition, the function u^h is differentiable only once, so, we cannot compute its second derivatives that form an operator L.

So, to apply this idea, we use the finite element solution u^h to construct a new smooth approximate solution $s \in S_2^3(\Omega)$. Informally speaking, this solution s must satisfy two conditions:

- First, it must be close to the original solution u^h; i.e., for every mesh point v_i, we must have $s(v_i) \approx u^h(v_i)$.

- Second, this function s must be an (approximate) solution of the equations (1). This means, in particular, that for every mesh point v_i, we must have $Ls(v_i) \approx f(x, s(v_i))$. The function f is non-linear, so, this condition is non-linear in s. However, we can linearize it if we take into consideration that $s(v_i) \approx u^h(v_i)$. Then, this condition can be reformulated as follows: $Ls(v_i) \approx f(v_i, u^h(v_i))$.

We cannot exactly satisfy both sets of conditions, so we must apply a techniques similar to least squares method to combine them. As a result, we arrive at the following method of determining s: For an arbitrary mesh point $x_0 = (x_1^0, x_2^0)$, we define the (local) polynomial solution p as a polynomial

$$p = \sum_{i,j:0\le i+j\le n} a_{i,j}(x_1 - x_1^0)^i (x_2 - x_2^0)^j$$

such that

$$\sum_{i=1}^{N} \alpha_i |p(v_i) - u^h(v_i)|^2 +$$

$$\sum_{i=1}^{M} \beta_i |Lp(w_i) - f(w_i, u^h(w_i))|^2 \to \min, \quad (6)$$

where $\{v_i\}_{i=1}^N$ and $\{w_i\}_{i=1}^M$ are nodes of the mesh,

$$\alpha_i = a_1/(\rho(v_i, x_0) + a_2),$$

$$\beta_i = b_1/(\rho(w_i, x_0) + b_2),$$

a_i and b_i are positive constants, and $\rho(x, y)$ denotes the distance between the points x, y.

The problem (6) is quadratic in the unknown coefficients a_{ij}. Thus, if we differentiate with respect to a_{ij}, and equate the derivatives to 0, we thus reduce this problem to the solution of the following system of linear algebraic equations:

$$Ba = d,$$

where

$$a = (a_{0,0}, a_{1,0}, a_{0,1}, \ldots, a_{n,0}, a_{n-1,1}, \ldots, a_{0,n}),$$

$$d = (u_1^h, \ldots, u_N^h,$$
$$f(w_1, u^h(w_1)), \ldots, f(w_M, u^h(w_M))).$$

After this smooth approximate solution s is computed, we can use the following defect [2]

$$\phi(x, s) = Ls - f(x, s), \quad x \in \Omega. \quad (7)$$

To get the desired interval estimate, we apply the finite element method (FEM) to solve numerically the following additional problem:

$$L_1 u_1 = 1, \quad x \text{ in } \Omega, \quad (8)$$

$$u_1(x) = 0, \quad x \text{ on } \partial\overline{\Omega}, \quad (9)$$

where $L_1 = Lu - qu$. We solve this auxiliary problem by constructing a special spline s_1.

Then, the desired interval solution has the form

$$\mathbf{u} = s + [\underline{\alpha}, \overline{\alpha}]s_1 + [\underline{\beta}, \overline{\beta}],$$

where

$$\overline{\alpha} = \max_{\overline{\Omega}}(\phi/(L_1 s_1), 0), \ \underline{\alpha} = \min_{\overline{\Omega}}(\phi/(L_1 s_1), 0),$$

$$\overline{\beta} = \max_{\partial\Omega}(-\overline{\alpha}s_1 - s, 0), \ \underline{\beta} = \min_{\partial\Omega}(-\underline{\alpha}s_1 - s, 0),$$

IV. How To Correct The Solution Of The Boundary Value Problem

To correct the solution, we must somehow find a good approximation for the difference $u - s$ between the exact (unknown) and the approximate solutions. To estimate this difference, let us find a differential equation that this function $\varepsilon = u - s$ must satisfy.

If we substitute $u = s + \varepsilon$ into the equation (1), we get the following equation:

$$Ls + L\varepsilon = f(x, s + \varepsilon),$$

or

$$L\varepsilon = f(x, s + \varepsilon) - Ls.$$

To estimate ε, we can esither solve this non-linear equation (using the same techniques as above), or simplify it. We can simplify this non-linear equation if we take into consideration the fact that ε is small and therefore, we can expand f into a power series in ε and keep only linear terms. As a result, we get the following equation:

$$L\varepsilon = f(x, s) + r(x)\varepsilon - Ls,$$

where we denoted

$$r(x) = \frac{\partial f}{\partial u}(x, s(x)).$$

If we move all the terms that contain u into the left-hand side, and all the other terms into the right-hand side, we get the following equation:

$$L\varepsilon - r(x)\varepsilon = \psi(x),$$

where $\psi(x) = f(x, s(x)) - (Ls)(x)$.

Let ε^h be a solution of this approximate equation, obtained by a similar finite-element technique. Then, for the corrected solution

$$s_{cor} = s + \varepsilon^h,$$

the following result is true:

THEOREM. *If $u \in W_2^4(\Omega)$, then*

$$\|u - s_{cor}\| \leq C \cdot h^4 \|u\|_{W_2^4(\Omega)},$$

where h is the mesh size, and C is a constant independent of h.

To get an even better solution, we can apply the correction procedure to the corrected solution; then, if necessary, we can apply the correction procedure again and again, and get better and better solutions.

In principle, instead of starting with an approximate solution, we can start with the $u^h = 0$ and $s = 0$. Thus, we will get a new iterative method of numerical solution for such equations.

References

[1] B. S. Dobronets, "Difference correction by finite elements of higher order", In: **Mathematical models and methods of solving the problems of continuous media**, Krasnoyarsk, 1986, pp. 80–88 (in Russian).

[2] B. S. Dobronets, V. V. Shaydurov, **Two-sided Numerical Methods**, Nauka Publ. (Siberian Branch), Novosibirsk, 1990 (in Russian).

[3] G. Streng and G. F. Fix, **An Analysis of the Finite Element Method**, Englewood Cliffs, Prentice-Hall, 1973.