On the Interval Multisplitting AOR (IMAOR) Method

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***Abstract*— Frommer and Mayer have developed an interval multi-splitting method for solving the systems of linear interval equations**

$$\mathbf{A}x = \mathbf{b},$$

where $\mathbf{A} \in I(R^{n\times n})$ is a regular interval matrix, and b is an interval vector. Based on this method, we develop a new method called *multispilitting AOR (IMAOR)*. The convergence of this method is discussed. For the case when the matrix A is an $H-$matrix, we formulate conditions that guarantee convergence. These conditions are easy to check.

Definition 1. An interval matrix $\mathbf{A} \in I(R^{n\times n})$ is called *regular* if each real matrix $A \in \mathbf{A}$ is nonsingular.

Definition 2. Let $\mathbf{A} \in I(R^{n\times n})$ be a regular interval matrix, and let $\mathbf{b} \in I(R^n)$ be an interval vector. Then, by *solving linear interval equations*

$$\mathbf{A}x = \mathbf{b}, \tag{1}$$

we mean the problem of finding an interval enclosure of the solution set

$$S = \{x \,|\, x = A^{-1}b, A \in \mathbf{A}, b \in \mathbf{b}.\}, \tag{2}$$

Comment. The goal is to find a *good* enclosure.

In order to compute the solution (1) iteratively, Frommer and Mayer [3] proposed a class of methods called *interval multisplitting methods* to enclose the solution set S given by (2). Let us explain this class of methods:

Denotation 1. By $IGA(\mathbf{M})(\mathbf{r})$, we will denote the result of applying interval Gaussian algorithm (IGA) to the equation $\mathbf{M}x = \mathbf{r}$.

Definition 3. [3] An *interval multisplitting* of an interval matrix $\mathbf{A}$ is defined as a finite collection of triples $(\mathbf{M}_k, \mathbf{N}_k, \mathbf{E}_k), k = 1, 2, ..., K$ of interval matrices $\mathbf{M}_k, \mathbf{N}_k, \mathbf{E}_k \in I(R^{n\times n})$, such that the matrix $\mathbf{M}_k$ if regular for all k, and the following three conditions are satisfied:

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1. For every k, $\mathbf{A} = \mathbf{M}_k - \mathbf{N}_k$.
2. For every k, the interval Gaussian algorithm IGA is feasible when applied to the equation $\mathbf{M}_k x = \mathbf{b}$ with an arbitrary right-hand side $\mathbf{b}$.
3. For every k, $\mathbf{E}_k$ is a diagonal matrix with non-negative entries. These matrices must satisfy the following equation:

$$\sum_{k=1}^{K} \mathbf{E}_k = I.$$

INTERVAL MULTISPLITTING METHOD. [3] *Suppose that we want to solve a problem (1), and that an interval multisplitting is given for the matrix* $\mathbf{A}$. *The corresponding interval multisplitting method to enclose the set S given by (2) is defined by the following iteration:*

$$\mathbf{x}^{(m+1)} = \sum_{k=1}^{K} \mathbf{E}_k \mathbf{y}^{m,k}, \; m = 0, 1, ..., \tag{3}$$

where

$$\mathbf{y}^{m,k} = IGA(\mathbf{M}_k)(\mathbf{N}_k \mathbf{x}^{(m)} + \mathbf{b}), \; k = 1, 2, ..., K.$$

Comment. This interval multisplitting method possesses a natural *parallelism*: the computations of $\mathbf{y}^{m,k}$ for different k are independent on each other, and therefore, these computations can be computed on K different processors of a multiprocessor system.

The fact that we do not not need to compute the i-th component of $\mathbf{y}^{m,k}$ if the corresponding diagonal entry $(\mathbf{E}_k)_{ii}$ of the matrix $\mathbf{E}_k$ is zero, makes these computations even faster. This may result in considerable savings of computational time.

Our idea. In this paper, we present a class of the interval multisplitting AOR (IMAOR) methods for solving large linear interval systems of equations (1). The convergence theorems for this new algorithm are established under the condition that the coefficient matrix $\mathbf{A}$ is an $H-$matrix. These convergence conditions are easy to check.

To formulate our algorithm, we will need to modify the definition of a multisplitting.

Definition 4. Let $\mathbf{A} \in I(R^{n\times n})$ be a regular interval matrix. By a *multisplitting* of $\mathbf{A}$, we will understand the finite collection of triples $(\mathbf{D}-\mathbf{L}_k, \mathbf{U}_k, \mathbf{E}_k)$, $k = 1, 2, ..., K$ of interval matrices, where:

- $\mathbf{D}$ is a diagonal part of the matrix $\mathbf{A}$;
- each matrix $\mathbf{L}_k$ is strictly lower triangular;
- each matrix $\mathbf{U}_k$ is strictly upper triangular,

for which the following three conditions are satisfied:

1. For every k, $\mathbf{A} = \mathbf{D} - \mathbf{L}_k - \mathbf{U}_k$.
2. There exists a real number γ such that for every k, the interval Gaussian algorithm IGA is feasible when applied to the equation $(\mathbf{D}-\gamma\mathbf{L}_k)x = \mathbf{b}$ with an arbitrary right-hand side $\mathbf{b}$.
3. For every k, $\mathbf{E}_k$ is a diagonal matrix with non-negative entries. These matrices must satisfy the following equation:

$$\sum_{k=1}^{K} \mathbf{E}_k = I.$$

Denotation 2. Suppose that we are given:

- a system (1),
- a multisplitting for the corresponding matrix $\mathbf{A}$, and
- a real number ω.

Then, we will define $f_k(\gamma, \omega, \mathbf{x})$ as

$$f_k(\gamma,\omega,\mathbf{x}) = IGA(\mathbf{D}-\gamma\mathbf{L}_k) \{[(1-\omega)\mathbf{D} + (\omega-\gamma)\mathbf{L}_k + \omega\mathbf{U}_k]\mathbf{x} + \omega\mathbf{b}\}.$$

IMAOR METHOD. *Start with an arbitrary vector* $\mathbf{x}^{(0)} \in I(R^n)$. *Then, for* $m = 0, 1, 2, ...,$ *compute*

$$\mathbf{x}^{(m+1)} = \sum_{k=1}^{K} \mathbf{E}_k f_k(\gamma, \omega, \mathbf{x}^{(m)}),$$

until this procedure converges.

Comments. This method can be reformulated as

$$\mathbf{x}^{(m+1)} = T_{IMAOR}(\omega,\gamma)\mathbf{x}^{(m)} + g_{IMAOR}(\omega,\gamma),$$

where we denoted

$$T_{IMAOR}(\gamma,\omega) = \sum_{k=1}^{K} \mathbf{E}_k \cdot IGA(\mathbf{D}-\gamma\mathbf{L}_k) [(1-\omega)\mathbf{D} + (\omega-\gamma)\mathbf{L}_k + \omega\mathbf{U}_k],$$

and

$$g_{IMAOR}(\gamma,\omega) = \sum_{k=1}^{K} \mathbf{E}_k \cdot IGA(\mathbf{D}-\gamma\mathbf{L}_k)(\omega\mathbf{b}).$$

If $K = 1$, and $\mathbf{A}$, $\mathbf{L}_1$, and $\mathbf{U}_1$ are point matrices, then the above IMAOR methods reduces to the AOR method proposed in [4].

Our main results are as follows:

THEOREM 1. *If* $\mathbf{A}$ *is an* $H-$*matrix,* $0 \le \gamma \le 1$, $0 < \omega < 1$, *and a multisplitting of* $\mathbf{A}$ *is given, then for an arbitrary starting vector* $\mathbf{x}^{(0)}$, *the following results are true:*

1) *IMAOR method is feasible, i.e., the sequence* $\mathbf{x}^{(m)}$ *converges to an interval vector* $\mathbf{x}^*$.
2) *IMAOR method is monotonic with respect to inclusion, i.e., if* $\mathbf{x}^{(0)} \subseteq \mathbf{y}^{(0)}$, *then* $\mathbf{x}^{(m)} \subseteq \mathbf{y}^{(m)}$ *for all* m.
3) *If* $S \subseteq \mathbf{x}^{(0)}$, *then* $S \subseteq \mathbf{x}^{(m)}$.
4) $S \subseteq \mathbf{x}^*$.

COROLLARY. *Suppose that the matrix* $\mathbf{A}$ *satisfies the following two conditions:*

- $\mathbf{A}$ *is an* $M-$*matrix;*
- $\mathbf{A}$ *is either strictly or irreducibly diagonally dominant.*

Suppose also that $0 \le \gamma \le 1$, $0 < \omega < 1$, *and a multisplitting of* $\mathbf{A}$ *is given. Then, for an arbitrary starting vector* $\mathbf{x}^{(0)}$, *the statements 1)-4) are true.*

References

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