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Data Processing Beyond Traditional Statistics: Applications of Interval Computations. A Brief Introduction

V. Kreinovich

***Abstract*— This paper contains a brief survey of the application described in this volume.**

35 years ago, computational operations with intervals were first introduced [44, 45]. They were not introduced in a theoretical journal, they appeared as Technical Reports of Lockheed Missile and Space Co., with the goal of solving applied real-life problems.

30 years ago, the first survey on interval computations have appeared [46]. It describes several different methods, and it also describes a potential application: to compute the trajectory of a spaceship going to the Moon.

Since then, many exciting applications have appeared. Some of them use only operations with intervals, some use other mathematical techniques as well. This survey is intended to provide a guide to the applications presented in this workshop.

I. Data Processing

One of the main functions of a computer is crunching numbers, or, to use a more fancy term, *data processing.* This data comes either from measurements, or from expert estimates, or from the results of the previous processing. Expert estimates are often very important, but a computer can easily crunch (and does crunch) thousands and millions numbers per second. There is not enough experts in the world to supply that many expert estimates. Therefore, the majority of data processing algorithms do not use expert estimates at all: they either directly process the results of the measurements, or process the results of pre-processing these results. In the second case, we can consider the entire chain of processors starting from the sensors and eventually producing the desired result as one huge data processing algorithm. So, in the majority of cases, *data processing processes the results of measurements.*

The author is with Computer Science Department, The University of Texas at El Paso, El Paso, TX 79968, email vladik@cs.utep.edu. This work was partially supported by NSF grant No. CDA-9015006 and NASA Research Grant No. 9-757.

II. Why Do We Process Data At All?

- First of all, we want to *know* the world that we live it. In particular, we want to know the characteristics of the physical quantities that characterize this world. In a few cases, we can simply measure them (e.g., body temperature, or weight). But for more complicated characteristics, like an amount of oil in the well, or a distance to a star, there is no way to measure them directly. So, we measure what we can and then, from the results of these measurements, we try to reconstruct the desired value.
- Second, we want to *change* this world: we want to control the objects that already exist, or to change them, or even to create (*synthesize*) new objects.

III. Main Problem

And here comes the problem. Measurements are never absolutely precise. The result $\tilde{x}$ of measuring a physical quantity x (e.g., temperature) may differ from the actual value of that quantity. E.g., if you have weighed yourself, and the result is 125 pounds, this does not mean that you weight equals exactly 125. If the scales have an accuracy ± 2, then the actual weight can be any number from 123(=125-2) to 127=(125+2).

So, the *data* that we process are *not* absolutely *precise.* This inaccuracy leads to the inaccuracy in the *result* of data processing. The problem is *to estimate the resulting inaccuracy.*

If we do not know this accuracy, then the result of data processing is of little practical use: e.g., suppose that we estimated that a given location contains 100 million tons of oil, but we do not know the accuracy of this estimate. Then, if this is 100 ± 5, this location is worth developing, but if it is 100 ± 100, then we need further analysis to decide what to do.

IV. In Many Cases, We Know Probabilities Of Errors

In many cases, the manufacturer of a measuring instrument provides us with the probabilities of different values of a measurement error. For such cases, there exist numerous methods that compute statistical characteristics of the resulting error (see, e.g., [55]).

V. In Many Cases, We Do Not Know Probabilities

In many other cases, however, the values of the probabilities are not known. Instead, the manufacturer provides us with the guaranteed accuracy Δ, i.e., with a guaranteed upper bound of the error $\Delta x = \tilde{x} - x$ (e.g., "error cannot exceed 0.1"). If our measurement results is $\tilde{x}$, then the possible values of $x = \tilde{x} - \Delta x$ form an *interval* $[\tilde{x} - \Delta, \tilde{x} + \Delta]$ ([24, 4, 55].

Historical comment. The first person to describe intervals as a result of measurement was Norbert Wiener: in [74], he applied intervals to measuring distances, and in [75], to measuring time.

Since we are dealing with intervals, the entire area is called *interval computations.*

In this case, our problem takes the following form:

VI. Main Problem (In Case We Do Not Know Probabilities)

Suppose that we are interested in the value of a physical quantity y that is difficult or impossible to measure directly. To find the value of y, we measure several other quantities $x_1, ..., x_n$ that are related to y, and then we reconstruct the value of y from the results $\tilde{x}_i$ of measuring x_i. In other words, we have an algorithm f that takes the values $\tilde{x}_i$ and returns an estimate $\tilde{y} = f(\tilde{x}_1, ..., \tilde{x}_n)$ (this estimate is called the *result of indirect measurement*).

In some cases, this algorithm consists of simply applying known formulas to $\tilde{x}_i$. In other cases, this algorithm implements a numerical method for solving a system of equations that connect x_i and y. These equations can be algebraic, differential, integral, etc, and the resulting algorithm can be pretty complicated.

The problem is to estimate the error of the estimate $\tilde{y}$:

We know:

- n intervals $\mathbf{x}_i = [\tilde{x}_i - \Delta_i, \tilde{x}_i + \Delta_i]$ and
- an algorithm f that transforms n real numbers $x_1, ..., x_n$ into a real number $y = f(x_1, ..., x_n)$.

We are interested in: estimating the interval

$$f(\mathbf{x}_1, ..., \mathbf{x}_n) = \{f(x_1, ..., x_n) \,|\, x_1 \in \mathbf{x}_1, ..., x_n \in \mathbf{x}_n\}.$$

This is the basic problem of interval computations with which the entire area started (see, e.g., [47]).

VII. Where Does This Main Problem Fit Into Mathematics

- For many algorithms f, methods of estimating the accuracy of the result $\tilde{y}$ have been developed in *numerical mathematics.* These methods often produce the *exact* error bounds.

  Numerical mathematics does not always give us the solution to our problem. Error estimation methods of numerical mathematics are often very complicated, and require difficult mathematical techniques. Because of that, algorithms f for which such methods are known form a small subset in the set of all algorithms that are used for data processing. New data processing problems appear every day, and new data processing algorithms are being designed to solve these problems. For a new data processing algorithm, for a reasonable period of time, no error estimation method is known. To handle these algorithms, we need a general tool that would, given f, $\tilde{x}_i$, and Δ_i, automatically estimate the error bound for y.

- Problems that appear when we do not know the exact distribution, but we know instead that a distribution belongs to a given *class* $\mathcal{P}$ of distributions, are called problems of *robust statistics* ([27]; [70], Ch. 16). So, from statistical viewpoint, interval computations is a particular case of robust statistics, in which $\mathcal{P}$ is the class of all distributions located on a given interval.

  Robust statistics does not always give us the solution to our problem because robust methods have been developed only for some functions f.

- The case when instead of a number, we have an interval (or, more generally, a set) of possible values, has been developed in mathematics under the name of *set-valued analysis* (see, e.g., [6]). In particular, differential equations with the uncertain (set-valued) right-hand side (called *differential inclusions*) are used, e.g., in control problems [5, 6].

  Set-valued analysis does not always give us the solution to our problem because its methods have been developed only for specific f, e.g., when f is a solution of a differential or an integral equation of a certain type.

VIII. Applications of the Main Problem

In this volume, the following applications are presented:

- to computing the *diameter of the laser beam* [59];
- to computing the stability times, in particular, for *particle accelerators* [11];
- to processing *geophysical data* [22, 48].

If we have several variables, then in addition to the intervals of possible values for each of them, we may have additional information about the relationship between x_i. As a result, the set of possible values of $(x_1, ..., x_n)$ forms a *convex set* (not necessarily a box). Applications of this approach to *civil engineering* problems are presented in [23].

Special computational problems appear when the results x_i of direct measurements are obtained from measuring one and the same quantity in the different moments of time. These problems are outlined in [77].

IX. Additional Problems

As we have already mentioned, if we cannot measure y directly, then we must do the following:

- first, find a *relationship* between the desired quantity y and the directly measurable quantities x_i;
- second, use this relationship to design an *algorithm* f that transforms the results $\tilde{x}_i$ of direct measurements into an estimate $\tilde{y}$ for y;
- third, use the known accuracies Δ_i of measuring x_i to estimate the *accuracy* Δ of the final estimate $\tilde{y}$.

The main problem describes the case when we already have an algorithm f. However, in some cases, we do not know f, and finding f is part of the problem. Depending on what we know, we can consider two different situations:

- We want to estimate the accuracy Δ of the result of data processing, but the relationship between the directly measurable quantities x_i and the desired quantity y is known only indirectly (not in the form of an algorithm).
- We want to estimate the accuracy Δ of the result of data processing, but the relationship between the directly measurable quantities x_i and the desired quantity y has to be determined from the experimental data (this is called an *identification problem*).

X. What If The Relationship Between The Directly Measured Quantities x_i And The Desired Quantity y Is Known Only Indirectly (Not In The Form Of An Algorithm)

In this case, if we want the results of our computations to be guaranteed, then we need some method like interval computations.

A. Explicit analytical expressions

Mathematically, the simplest case is when we have an *explicit analytical expression* for the dependency of y on x_i (but computationally, we still have a problem). For this case, a computation method has been proposed in [34].

B. Equations and systems of equations

A more complicated case is when we have an expression for the desired function as a *sum of an infinite series* (it can be Fourier, Taylor series, etc). This case is described in [64].

Instead of an explicit expression, we may have an *equation* or a *system of equations* that describes the desired dependency:

- It can be a system of *non-linear (algebraic) equations*; an application for computer graphics is described in [13].
- It can be an *ordinary differential equation* (ODE), or a system of ODE. This case is analyzed:
  - in [80, 81], with applications to *airplane inertial navigation*, and
  - in [43], to compute a singularity point, with possible applications to equations that emerge in *optimal control* theory.
- It can be a *partial differential equation* [21].
- It can be an *integral equation* [14], with an application to *astrophysics*.
- It can be a complicated equation described in terms of *sets* [65], with an application to the description of a *milling surface* and other *manufacturing processes*.

C. Systems of equations and inequalities

Instead of a system of equations, we may have a *system of equations and inequalities* (constraints). This case is considered in [28, 29, 58].

D. Optimization problems

The relationship between the known values x_i and the desired value y can also be described in terms of some *optimization problem*. In this case, we must get a guaranteed solution of the optimization problem. A survey of such methods is presented in [31].

In general, the global optimization problem is NP-hard (i.e., crudely speaking, every algorithm requires exponential time on at least some of the cases). Therefore, these algorithms tend to take (sometimes) too long to compute. Several approaches have been used to speed up these computations:

- *neural networks* [49] (with an application to *traffic control*);
- *genetic algorithms* [2, 52];
- *simulated annealing* [52];
- *design of experiments* [2];
- *logic programming* [16, 28, 58].

The combination of constraints and optimization is analyzed in [16, 28, 58].

XI. What If The Relationship Between Direct Measurements And The Desired Value Has To Be Determined? Identification Problem

A. General Case

An application to determining the parameters of *learning curves* (that describe how people learn manufacturing skills) is given in [10].

An application to *quality control* in manufacturing is described in [26].

An application to finding the parameters of *time series* is described in [12].

In general, the interval identification problem is also NP-hard, so, heuristic methods are used: e.g., *neural networks* are used in [54].

B. Linear Models

B.1. Why Linear Models

If we cannot reconstruct the *exact* dependency, then, maybe, we will be able to neglect some small terms, and solve the resulting simplified system. For example, if the values x_i are small, then we can neglect quadratic and higher order terms in the expansion of the function f, and approximate y with a linear dependency: $y \approx y_0 + y_1 x_1 + ... + y_n x_n$. In this case, after N measurements, we get intervals $\mathbf{y}^{(k)}$ and $\mathbf{x}_i^{(k)}$, $1 \le k \le N$, of possible values of y and x_i, and *identification problem* means that we must find the values y_i that satisfy the following system of equations:

$$\mathbf{y}^{(k)} \approx y_0 + y_1 \mathbf{x}_1^{(k)} + ... + y_n \mathbf{x}_n^{(k)}, \ 1 \le k \le N. \quad (1)$$

In mathematical terms, we must find the values y_i for which for each k, the corresponding values can be equal (i.e., for each k, the left- and the right-hand sides of the equation (1) have a common point). In other words, we arrive at a problem of solving a *system of interval linear algebraic equations.*

B.2. Two Possible Formulations Of Identification Problems

From the computational viewpoint, this problem can be formulated in two different ways:

- To find *some* values y_i that are consistent with the measurements (in the above sense).
- To find *all* values y_i that are consistent with the given measurements.

We may also have a situation in which each measurement result

$$(\mathbf{y}^{(k)}, \mathbf{x}_1^{(k)}, ..., \mathbf{x}_n^{(k)})$$

may correspond not to a *single* observation, but to a *series* of observations, and may mean that every time when the inputs were in $\mathbf{x}_i^{(k)}$, the results were in $\mathbf{y}^{(k)}$. In this case, $\approx$ in the above formula means $\supseteq$. Different formulation are described and compared in [60].

B.3. Even For Linear Models, Identification Problem Is In General Intractable

Even for linear systems, the problem is still NP-hard [38, 56].

B.4. Efficient Algorithms For Particular Cases Of Linear Models

For some particular cases of linear systems

$$\mathbf{A}x = \mathbf{b},$$

efficient algorithms are possible:

- algorithms for computing an *inner estimation* for the solution are proposed and described in [35, 38, 60]; these algorithms are based on using *Kaucher's arithmetic* (a generalization of interval arithmetic that allows intervals $[a^-, a^+]$ with $a^- > a^+$);
- for *symmetric* matrices, algorithms are described in $\mathbf{A}$ [3];
- for *$H-$matrices*, a new method is described in [15].
- If input intervals are small enough, then there exists a feasible algorithm that computes the exact solution in it almost all cases [39].

C. If Linear Models Do Not Work, Which Non-Linear Models Should We Choose?

For the cases when linear dependency does not work, [67] advocates the use of *fractionally linear* models.

D. Physics As Identification

If we interpret the word "identification" broadly enough, as determining the (unknown) dependency between physical quantities, then the entire *theoretical physics* becomes a particular case of an identification. In [33], it is shown that if we take measurement uncertainty into consideration, then some divergence problems will disappear.

XII. Synthesis, Design, Control, And Decision Making

In the previous sections, we have described applications to *knowing* the world. Let us now describe the related problems that appear when we want to *change* it, i.e., problems of decision-making, design, and control.

A. Goal: find an element from a finite set

The simplest (at least in formulation) decision-making problems are when we must find all elements from a given finite set that satisfy certain conditions. Two problems of this type are described in this volume:

- finding input variables whose influence on the result is the largest [9], with the application to detecting defective stages in VLSI manufacturing;
- locating local extrema of a function of one variable from interval measurement results [69], with applications to *chemical spectral analysis*, *image processing in radioastronomy*, *elementary particles*, and *banking* [20].

B. Goal: find the values of finitely many parameters

The next in complexity are problems in which the goal is to find one or several continuous parameters. Such applications include:

- making decisions in *economy* [20, 30], where interval uncertainty has to be taken into account;
- *optimal design* in manufacturing [26].

C. Goal: find a function

Finally, the most complicated problems are the ones in which one must find a *function*: it may be the dependence of control parameters on the input (i.e., *control strategy*), or the dependence of some parameter on time, etc. The examples of these applications include:

- control of *mobile robot* [76];
- *congestion control of computer networks* [66];
- *robust stable* control [7];
- multi-input multi-output control [1], with applications to the control of *airplane engines* and *power plants*.

For control, optimization problems are often NP-hard [63] (even for linear systems with interval uncertainty), so, in hard cases, it may be necessary to use *expert knowledge* to produce the good control.

D. Holistic approach to design

In real-life, mathematically formulated design problems (that we have been talking about) and more down-to-Earth problems (like: should we choose an 8-bit or a 16-bit processor, and which of the three engines on the market should we buy for our robot) are often solved together. This idea is described in general terms in [25]. A practical example of this general design approach is given in [8].

XIII. Beyond Intervals

Interval data processing methods have been designed for the case when the only information we have about the error of a direct measurement is the guaranteed upper bound Δ for this error. In this case, if the result of measuring a quantity x is $\tilde{x}$, then the only thing that we know about the actual value of x is that this x belongs to an *interval* $[\tilde{x}-\Delta,\tilde{x}+\Delta]$.

In many cases, we have an *additional* information about the error. This additional information can include:

- the probabilities of different values of error $\Delta x = \tilde{x} - x$ (i.e., *statistical information*), and/or
- *expert's knowledge* of the possibility of different values of Δx.

In many of these cases, interval methods are still useful. Some of these cases are described in the present volume.

XIV. Applications To Statistics

Depending on how much knowledge we have about the probabilities, we can distinguish between two types of situations:

- In some cases, we have enough information to determine the type of the distribution (e.g., we know that it is a Gaussian distribution), and we know the approximate values of the *parameters* of this distribution. Statistical methods (based on finite sample) are never absolutely precise, and besides, the observations on which these estimates are made are also not precise. Therefore, instead of the precise values of the parameters, we only have *intervals* that are guaranteed to contain these values. In this case, if we want to predict the values of the related parameters, and we want to *guarantee* the results of the computations, then we must use *interval computations*. Such cases are described in [71, 73].

In particular, for *queuing models*, this approach is applied in [18] to:

  - *communication networks*;
  - *management of the engineering team projects*;
  - *transportation systems*;
  - *flexible manufacturing systems*;
  - *service systems*;

- In some cases, we do not have enough information to determine the family of distributions. Instead, we have *estimates* of certain characteristics (e.g., of the moments, and/or of the distribution function). Since these estimates come from observations, they are not precise: e.g., we usually have an *interval* of possible values of the moment. In this case, if we want to know some other statistical characteristics of the same variable, or statistical characteristics of the related variables (e.g., of $f(x_1, ..., x_n)$ when we have some information about x_i), then, we will also come up with *intervals* of possible values. Such methods are described in [17, 36, 37, 61, 62].
- In some cases [51], natural requirements for a characteristic are inconsistent if we are looking for a numerical one, but there is an *interval-valued* characteristic that satisfies all desired requirements. In [51], applications to *sociology* and to *quality control* are described.

XV. Applications Of Interval Methods To Expert Knowledge

All the information about the real world comes from two sources:

- from measurements, and
- from experts.
  - Expert knowledge is often *needed* because without such a knowledge, e.g., the problems of optimal control are intractable [63].
  - Moreover, the type of knowledge provided by the experts is *optimal* (in some reasonable sense) [40]. So, in this sense, expert knowledge is *the best* way to handle such situations.

We already know that interval methods are useful in processing measurements data. In many cases, they are also useful in processing expert's knowledge as well.

Experts' knowledge is extremely important in many fields like medicine, geology, economy (and even engineering), where we (often) do not have enough confirmed mathematically precise models, but we still need to make decisions.

In order to describe different types of expert's knowledge (as opposed to mathematically precise knowledge), let us briefly recall how precise knowledge is described. This knowledge can be described by formulas that are obtained from *elementary formulas* by using logical connectives $\vee, \&$, etc, and quantifiers $\forall x$ and $\exists x$. Since we are talking about the mathematically precise knowledge, each formula is either true or false.

Expert's statements are rarely describable in these terms, because, the experts are often not 100% sure about their statements. They phrase it like "If a patient has a fever, and a sore throat, and a headache, then most probably, he has a fever".

Interval methods can be used to process expert knowledge:

- A natural idea is to describe a *set* of possible degrees of belief to describe all the intricacies of human beliefs. In this case, to describe the expert knowledge about a certain value x, instead of a single interval of possible values of x, we have *several* intervals that corresponds to different degrees of possibility. This is a much more adequate description of human knowledge [42].
- With many different degrees of belief, we encounter the following additional problem: an expert may be unable to precisely describe his degree of belief. Therefore, instead of a single value that describes his degree of belief, we will have an *interval* of possible values of degree of belief [50]. Even if we have the exact values of the expert's degree of belief in basic statements E_i from the knowledge base, still, we will have interval uncertainty if we are looking for the degrees of belief in the queries that may be logical combinations of these statements E_i:
  - The degree of belief $d(A\&B)$ in, say, $A\&B$ is not uniquely determined by the degrees of belief $d(A)$ and $d(B)$ in A and B: if the statements A and B are independent, then we have one result, and if they are correlated, we have another one. So, if we only know the degrees of belief in A and B, then we have an *interval* of possible values of degree of belief in $A\&B$ [32].
  - Even if we fix a function that transform $d(A)$ and $d(B)$ into a reasonable estimate for $d(A\&B)$, still, for more complicated expressions, we will have an additional problem caused by the fact that every expression, can be described in several dif-

ferent ways in terms of the basic logical operations &, ∨, and ¬ (CNF form, DNF form, etc). These expressions are equivalent in 2–valued logic, but if we use these expression to compute degrees of belief, we end up with different results. So, for such an expression F, instead of the *exact* value of $d(F)$, we end up with an *interval* of possible values of $d(F)$ [68, 79].

If we know the intervals that represent the degrees of belief in A and B, how can we estimate the degree of belief in $A\&B$? This question is analyzed in [78].

XVI. Hardware And Software Support For Interval Computations

Algorithms that provide *guaranteed estimates* for the results of data processing are usually more complicated than traditional data processing algorithm that simply provide us with an estimate (and give no bound for this estimate). Since interval algorithms are more complicated, they take much longer to compute than traditional ones. In real-life situations, however, we are often under serious time pressure, and we cannot afford to spend any extra computation time. Therefore, we must *speed up* interval computations. For that, we need special hardware and software support [72].

A. Hardware support

- The first natural idea is to use hardware speed-up that is already possible in the existing computers:
  - *parallelism* [15, 48];
  - hardware supported *operations with long machine words* [19].
- Another possibility is to design *special hardware* [57].

B. Software support

To provide software support for interval computations, we may use software ideas that have already been shown to be useful in computing:

- *functional programming*, that enables to prove correctness of the programs [41];
- *logic programming*, that enables us to describe the *goal* of the program without having to always specify the algorithm [16];
- *bag languages*, that describe the sequence of operations in an easy-to-parallelize way [19];
- *genetic programming*, that enables us to design a program by directed stochastic search (that uses the ideas from the biological evolution) [53].

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