Verified Integration and Generation of Poincare Maps Using Taylor Models

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The field of nonlinear dynamics is simultaneously one of the oldest and one of the newest areas of research in mathematical physics; originating in the study of planetary motion and the stability of the solar system, it has experienced a renaissance in many scientific disciplines. One of the many currently important problems is the study of long term stability of motion in large particle accelerators, which formally can be characterized by moderately nonlinear differential equations also characteristic of planetary motion.

In case nonlinear dynamics is governed by differential equations, it is often described by the so-called Poincare map which expresses the functional dependence of final coordinates on initial coordinates for a discrete step in the independent variable. Using nonlinear normal form methods [1], we have recently shown [2] how a Poincare map can be used to extract guaranteed bounds for stability times of dynamical systems. In particular, the methods were applied to the study of dynamics in particle accelerators for several practical cases [3].

The substantial complexity associated with the practical use of the method, reflected in execution times of hours to days, requires careful control of blow up as well as efficient strategies to perform verified six-dimensional optimization. For both of these problems it proved necessary to depart from plain interval arithmetic and to represent functional dependencies by Taylor polynomials with interval remainder bounds. These so-called Taylor models [4] represent a hybrid of high-order multivariate automatic differentiation tools [5] to obtain the Taylor polynomials and interval methods to bound the Taylor remainder.

The rigor associated with the method requires a rigorous knowledge of the Poincare map. Furthermore, any error bounds for it have to be tightly controlled since they inversely proportionally translate into the stability times that can be guaranteed with the normal form methods. To determine the value of the Poincare map at individual initial conditions requires the use of verified numerical integrators. We will show how Taylor model methods can be combined with Banach fixed point and differential algebraic techniques to obtain verified solutions of differential equations. The error bounds are obtained directly from within the method and do not rely on conventional estimates requiring the separate bounding of higher derivatives.

Since the Poincare map does not only represent the functional dependence of coordinates on time but also on initial conditions, it is necessary to extend the integration methods in such a way as to describe this dependence in a functional form. This requirement in particular is conveniently achieved with the use of Taylor models.

Applications of the method to complex practical problems in four and six dimensions from the field of accelerator physics will be given.

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