# Interval Approach to Learning Curves

Mario Beruvides and Vladik Kreinovich

## I. What is a Learning Curve

In order to make reasonable manufacturing decisions, we must be able to predict the workers' productivity in producing different objects.

Of course, we can ask several workers to perform the desired task and thus measure their current productivity. However, this measured productivity is a very good estimate for the future productivity: the workers' performance improves as they produce more and more units; in other words, they *learn*. To predict the performance after a certain amount of units, we must measure the performance at different moments of time, and then extrapolate the resulting dependency $t(n)$ of the time $t$ that is required to produce a single unit on the total (cumulative) amount $n$ of the units produced. The functions that are used to model this dependency are called *learning curves* (see, e.g., [3, 2] and reference therein).

Usually, the *hyperbolic* dependency is used: $t(n) = a \cdot n^\alpha$ with unknown $a$ and $\alpha$. This dependency can be further simplified if we move to a logarithmic scale: $T = \alpha N + A$, where $T = \ln(t)$, $N = \ln(n)$, and $a = \ln(A)$. So, to estimate $A$ and $\alpha$ from the experimental data $(n_i, t_i)$, $1 \le i \le K$, we can:

- compute the logarithms $T_i = \ln(t_i)$ and $N_i = \ln(n_i)$;
- find the $\alpha$ and $A$ from the equations $T_i = \alpha N_i + A$; and
- compute $a = \exp(A)$.

## II. Problems with Learning Curves

The dependency represented by a learning curve is usually only approximate.

Traditionally, statistical methods are used to describe the approximate character of this curve. However, it is well known (see, e.g., [1]) that the difference between the model and the actual productivity is not of purely stochastic nature. It is more of a bias than of a random component. A more adequate description of this difference is that it belongs to a certain *interval* $[-\Delta, \Delta]$, and that we know nothing about the probabilities of different values from this interval.

## III. Our Idea

In view of this description, it is very natural to apply *interval computations* to learning curves. Namely, if we have the measurement results $t_i$ and $n_i$, and the accuracy $\Delta$, then we must find $a$ and $\alpha$ from the conditions that $t_i - \Delta \le a \cdot n_i^\alpha \le t_i + \Delta$. In logarithmic scale, these conditions take the form:

$$\ln(t_i - \Delta) \le \alpha N_i + A \le \ln(t_i + \Delta).$$

Using linear programming, we can find the smallest and the largest possible values of $A$ and $\alpha$ that satisfy these inequalities. In other words, we will have the *intervals* $\mathbf{A} = [A^-, A^+]$ and $[\alpha^-, \alpha^+]$ that are guaranteed to contain the actual values of these parameters. From the interval for $A$, we can find an interval for $a$: $\mathbf{a} = [a^-, a^+]$, where $a^\pm = \exp(A^\pm)$.

Now, if we are interested in the value of $t(n)$ by the time when we have already produced $n$ units, we get an *interval* of possible values:

$$t(n) \in [a^- \cdot n^{\alpha^-}, a^+ \cdot n^{\alpha^+}].$$

This interval must be used for decision making: The cautious estimate should be based on the largest possible time, the most risky ones can use the fastest possible time.

## References

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[2] J. C. Martin, **Labor productivity control. New approaches for industrial engineers and managers**, Praeger, NY, 1991.

[3] B. W. Niebel, **Motion and time study**, R. D. Irwin, Homewood, IL, 1976.

The author is with the [1]Department of Industrial Engineering, College of Engineering, Texas Tech University, Lubbock, TX 79409, phone (806) 742-3543 (Dept.), email mberuvides@coe3.coe.ttu.edu, and [2]Department of Computer Science, Univeristy of Texas at El Paso, El Paso, TX 79968, email vladik@cs.utep.edu.