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Planar Geometry Problems Motivated by Robustness of Linear Systems

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Over the last decade, there has been a resurgence of research on robustness analysis for linear systems whose mathematical models include uncertain parameters. No statistical description for these parameters is assumed — only interval bounds are given. This lecture will focus on a number of planar geometry problems which arise in this context. In some cases, "clean" polynomial-time solutions can be given; in other cases, exponential-time solutions are available but become "impractical" to implement as the number of uncertain parameters increases. This leads to a number of fundamental questions about computational complexity; e.g., see [1].

Stability of a linear feedback system is determined by the roots of its characteristic polynomial. If this polynomial involves system parameters known only within given bounds, the classical robust stability problem arises. One approach to the solution of this problem involves generation of the so-called *value set*. This set lies in the complex plane and is a function of the frequency variable $\omega \geq 0$. The ability to generate the value set in graphics makes it possible to solve robustness problems in a computer-aided context.

To be more specific, the notation q is used to denote a vector of real uncertain parameters and $p(s,q)$ is assumed to be a polynomial having coefficients depending continuously on q. Hence, we can write

$$p(s,q) = s^n + a_{n-1}(q)s^{n-1} + \cdots + a_1(q)s + a_0(q)$$

with each $a_i(q)$ being a real continuous function. The given data also consists of lower and upper bounds for each component q_i of q. Hence, we use the notation Q for the box describing the domain of q.

Now, given a fixed frequency $\omega \geq 0$, we set $s = j\omega$ and formally define the *value set* $p(j\omega, Q) = \{p(j\omega, q) : q \in Q\}$. Note that $p(j\omega, Q)$ is the image of the multi-dimensional box Q under the mapping $p(j\omega, \cdot)$. The importance of the value set is derived from the following fact: If $p(s, q^0)$ is stable (all roots having negative real part) for at least one member $q^0 \in Q$, then robust stability (stability for all $q \in Q$) is guaranteed if and only if the *Zero Exclusion Condition* $0 \notin p(j\omega, Q)$ is satisfied for all $\omega \geq 0$. This type of stability condition goes back quite far in the literature; for example, see [2] for historical purposes and [3] for a more recent exposition.

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The first part of this lecture will be a review of those cases for which the value set generation problem has already been solved. For example, if each component of q enters into only one coefficient of $p(s,q)$, then Kharitonov's interval polynomial result [4] emerges as a consequence of value set rectangularity. For the case when the coefficients $a_i(q)$ are affine linear functions, the value set is readily shown to be a convex polygon. This leads to the Edge Theorem [5]. Finally, for the more general case when q enters nonlinearly, only very limited results are available. In this regard, perhaps the strongest result to date is the Mapping Theorem: For the case of multilinear dependence on the uncertain parameters, this theorem provides a simple recipe for generation of the convex hull of the value set; see [6]. Namely, this convex hull is the two-dimensional convex polygon whose vertices are obtained by evaluation of $p(j\omega, q)$ with q restricted to the vertices of the bounding box Q. More precisely, denoting the set of vertices of Q by $\{q^1, q^2, \ldots, q^L\}$, the Mapping Theorem indicates that

$$\text{conv } p(j\omega, Q) =$$

$$\text{conv}\{p(j\omega, q^1), p(\omega, q^2), \ldots, p(j\omega, q^L)\}.$$

The second part of this lecture will concentrate on new results obtained in collaboration with Tempo; see [7]. A *Generalized Mapping Theorem* will be presented and it will be argued that this theorem enhances the class of nonlinear dependencies for which the convex hull can be generated. The point of view taken in this work is that the value set generation problem is ill-posed unless computability considerations are taken into account. This leads to consideration of the current work in light of existing results on computational complexity.

References

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