# Robust Design of Control System for Multi-input Multi-output Plant with Interval Uncertainty

F. M. Akhmedjanov, V. G. Krymsky

## I. Formulation of the Problem

Recent widespread interest in robust design of control systems subject to parameter perturbations caused a considerable part of the research activity orientation toward the methods based on interval technique. These methods are designed to analyze systems with *interval uncertainty*, i.e., systems for which we do not know the exact values of some parameters in the their description, only intervals of possible values of these parameters [1]. The use of these methods allows us to obtain important results regarding stability and control quality provision for systems with interval uncertainty.

The majority of the current results correspond to the case when we already know the characteristics of the closed-loop system (CLS). For example, this is the assumption behind the famous Kharitonov's theorems [2]. In the real world, however, we only know the interval uncertainty in the description of the plant itself. So, to apply methods based on characteristics of the CLS, we must compute the intervals characteristics of the CLS from the interval characteristics of the plant.

These computations can be performed directly: by applying naive interval computations (or a similar technique) to the corresponding formulas. However, this approach has two major drawbacks:

- The corresponding computations often take too much time.
- This application of naive interval computations often leads to an overestimation of the interval characteristics of CLS. As a result, we may discard perfectly good control because, unlike the actual intervals, the overestimated intervals may lie within the required bounds.

In this paper, we describe better interval techniques.

## II. Mathematical Description of the Problem

Let us consider a linear time-invariant closed-loop system with multi-input multi-output (MIMO) dynamical plant. For such a plant, CLS behavior can be described by the equations

$$\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{R}(s)\mathbf{E}(s), \quad \mathbf{E}(s) = \mathbf{G}(s) - \mathbf{Y}(s), \quad (1)$$

where:

- $\mathbf{Y}(s) = \|Y_i(s)\|_{N\times 1},$, $\mathbf{G}(s) = \|G_i(s)\|_{N\times 1}$, and $\mathbf{E}(s) = \|E_i(s)\|_{N\times 1}$ are Laplace transforms of the actual output, of the desired output, and of the difference between them (called *control error*);
- $\mathbf{H}(s) = \|H_{ij}(s)\|_{N\times N}$ and $\mathbf{R}(s) = \|R_{ij}(s)\|_{N\times N}$ are $N \times N$ transfer matrices of the plant and of the controller;
- $s$ is a Laplace transform variable.

Each of the functions $H_{ij}(s), i, j \in \{\overline{1, N}\}$, is a ratio of two interval polynomials, i.e., polynomials whose coefficients are known only with interval uncertainty; in mathematical terms, the coefficients of these polynomials belong to the set of all interval numbers

$$\mathbb{I}(\mathbb{R}^+) = \{x \,|\underline{x} \le x \le \overline{x}, \ 0 < \underline{x} \le \overline{x}\}.$$

So, for each coefficient $\alpha_l$, only its bounds $\underline{\alpha}_l, \overline{\alpha}_l$ of $\alpha_l, l \in \{\overline{1, k}\}$ are known.

The goal of robust design is to find controller transfer matrix $\mathbf{R}(\mathbf{s})$ that would provides the desired performance of CLS for all possible combinations of MIMO plant parameters $\alpha_l, l \in \{\overline{1, k}\}$, from the given intervals. Typical examples of performance criteria include:

- system stability and
- required level of control quality.

## III. Two Approaches to Solving The Robust Control Problem

There are two basic techniques for solving traditional control problems:

- *Algebraic-polynomial* techniques that is based on Laplace transforms.
- *Frequency domain* methods that are based on Fourier transforms.

The authors are with the Ufa State Aviation Engineering University, 12 K.Marx Street, Ufa, Bashkortostan, 450000 Russia, email root@conver.bashkiria.su.

Similar methods can be applied to design of robust controllers. These methods will be described in the following two sections.

## IV. Algebraic Polynomial Approach

The corresponding method consists of the four stages:

### A. The first stage of the proposed method

The goal of the beginning (structural) stage is to choose the orders of the polynomials that are numerators and denominators of $R_{ij}(s)$, $i,j \in \{\overline{1,N}\}$. To find these orders, we formulate inequalities that reflect the requirement that the controlled plant must satisfy, such as:

- the possibility to physically implement the controller;
- dynamic accuracy;
- structural stability.

The interconnections between subsystems of the controller subsystems are also taken into account on this stage.

As an example, let us consider the conditions that describe structural stability of CLS. For this example, we will need the following notations:

- For a rational function $F$, by $m[F]$, we will denote the order of its numerator.
- For a rational function $F$, by $n[F]$, we will denote the order of its numerator.
- For a polynomial $P$, by $deg[P]$, we will denote its order.

Let

$$\mathbf{W}(s) = \mathbf{H}(s)\mathbf{R}(s) = \mathbf{Q}(s)/[s^\sigma \cdot T(s) \cdot Z(s)] \qquad (2)$$

be an open-loop system transfer matrix. Here,

- $s^\sigma \cdot T(s)$ is the common denominator for $R_{ij}(s)$, $i,j \in \{\overline{1,N}\}$;
- $Z(s)$ is the common denominator for $H_{ij}(s)$, $i,j \in \{\overline{1,N}\}$;
- $\mathbf{Q}(s)$ is a polynomial matrix;
- $\sigma$ is a positive integer number which is chosen to provide the desired control accuracy.

Then characteristic polynomial $B(s)$ of the CLS can be described by an expression

$$B(s) = \sum_{l=0}^{N} P_l(s), \qquad (3)$$

where:

- $$P_0(s) = [s^\sigma \cdot T(s) \cdot Z(s)]^N, \qquad (4)$$
- $$P_l(s) = D_l(\mathbf{Q}) \cdot [s^\sigma \cdot T(s) \cdot Z(s)]^{N-l}, \qquad (5)$$
- $D_l(\mathbf{Q})$ denotes a sum of all principle minors of $l-$th order of the matrix $\mathbf{Q}(s)$.

To obtain structural stability of CLS, it is necessary for $B(s)$ to contain all degrees of $s$ from zero to maximal value. Using (3), (4), and (5) one can form the following inequality that describes this requirement:

$$m[R_{j^*i^*}] \geq \sigma + deg[Z] - m[H_{i^*j^*}] - 1, \qquad (6)$$

where $i^*, j^*$ are such that

$$m[H_{i^*j^*}] = \max\{m[H_{ij}]\}. \qquad (7)$$

Inequality (6) thus describes the requirement of structural stability.

In [3], we describe how to find the values $m[R_{ij}], n[R_{ij}]$, $i,j \in \{\overline{1,N}\}$ that satisfy all inequalities (corresponding to all the requirements). Namely, we form a cost function that describes how the cost of the system depends on the choice of the degrees, and we look for the values that minimize this (linear) cost function under the condition that all inequalities are satisfied.

### B. The second stage of the proposed method

On the second stage, we compute the (interval) coefficients $b_0, ..., b_n$ of the characteristic polynomial $B(s)$ of CLS. This ($n-$th order) polynomial is defined as a numerator of the expression

$$\det[\mathbf{I} + \mathbf{H}(\mathrm{s})\mathbf{R}(\mathrm{s})], \qquad (8)$$

where $\mathbf{I}$ is the $(N \times N)$ identity matrix.

To following idea enables us to simplify (and thus, to speed up) the computations of these coefficients $b_i$: the vector $\mathbf{b^0} = ||\mathbf{b_i}||_{((\mathrm{N}+1)\times 1}$ satisfies the equation

$$F \cdot \mathbf{b^0} = \mathbf{B^0}, \qquad (9)$$

where:

- $\mathbf{B^0}$ is an $(n+1)-$dimensional vector of that consists of values $B(s_1), ..., B(s_{n+1})$ computed (using (8)) for positive real numbers $s_1, ..., s_{n+1}$, and
- $F = ||F_{ij}||_{(n+1)\times(n+1)}$ is a matrix with elements

$$F_{ij} = s_i^{n+1-j}, \quad i,j \in \{\overline{1,n+1}\}. \qquad (10)$$

Algorithms for solving the system (9) are described in [4].

### C. *The third stage of the proposed method*

#### C.1. *The main goal of the third stage*

The third stage of any algebraic-polynomial method is usually connected with placing the roots of $B(s)$ in the desired domain (since we want a stable control, this domain must lie in the left half of the complex plane). Frequently, the desired domain has the following trapezoid shape:

- its upper and lower bounds pass through 0, and form angles $\pm\varphi$ with the real axis;
- the right bound is located at a distance of $\eta$ from the imaginary axis.

Here, $\eta$ (called *stability degree*) and $\varphi$ characterize settling time and damping ratio of CLS dynamic processes. For such domains, checking whether all the roots in the domain, is equivalent to checking two things:

- Checking whether all the roots are to the left of the line $Re(z) = -\eta$.
- Checking whether all the roots are in the sector formed by the angles $\pm\varphi$.

#### C.2. *How is $\eta-$checking implemented in the existing interval methods*

In the existing interval control methods, to check whether the roots lie to the left of the desired line $\eta$, the following method is applied:

- First, we substitute $s = z - \eta$ into $B(s)$, and get a new polynomial $B^*(z)$. We use the known interval bounds of the coefficients of $B(s)$ to compute the intervals of possible values of the coefficients of $B^*(z)$.
- Then, we use the existing methods of checking stability of interval polynomials (e.g., Kharitonov's method) to check whether this polynomial $B^*(z)$ is stable.

If this polynomial is stable, then the $\eta-$condition is satifsied for all the roots of the original polynomial $B(s)$.

#### C.3. *The existing $\eta-$checking method often reduces the design accuracy*

The main problem with the above-described method is as follows: the computation of the interval enclosures for the coefficients of $B^*(z)$ include several interval operations. As a result, we get intervals that are guaranteed to contain the desired coefficients, but these intervals are an overestimate. So, to guarantee that these overestimated intervals are in the desired domain, we must impose much stricter requirements on the control than is really necessary. As a result, we reduce the accuracy of the design.

#### C.4. *$\eta-$checking: our idea*

To prevent the reduction of design accuracy, we propose a new method of $\eta-$checking that is based on checking a finite number of polynomials.

#### C.5. *$\varphi-$checking*

For $\varphi-$checking, we use algorithms proposed in [5]. In particular, the authors of [5] has shown that if $\varphi = (p/q)\pi$, where $p, q$ have no common divisors, then in order to be guaranteed that all roots of all (real-valued) polynomials from a given interval polynomial satisfy are inside the $\varphi-$sector, it is sufficient to check $2q$ (real-valued) polynomials. For example, for $p/q = 2/3$, it is sufficient to check the following polynomials:

$$B_1^{**}(s) = \overline{b}_0 + \overline{b}_1 s + \underline{b}_2 s^2 + \overline{b}_3 s^3 + ...,$$

$$B_2^{**}(s) = \overline{b}_0 + \underline{b}_1 s + \underline{b}_2 s^2 + \overline{b}_3 s^3 + ...,$$

$$B_3^{**}(s) = \overline{b}_0 + \underline{b}_1 s + \overline{b}_2 s^2 + \overline{b}_3 s^3 + ...,$$

$$B_4^{**}(s) = \underline{b}_0 + \underline{b}_1 s + \overline{b}_2 s^2 + \underline{b}_3 s^3 + ...,$$

$$B_5^{**}(s) = \underline{b}_0 + \overline{b}_1 s + \overline{b}_2 s^2 + \underline{b}_3 s^3 + ...,$$

$$B_6^{**}(s) = \underline{b}_0 + \overline{b}_1 s + \underline{b}_2 s^2 + \underline{b}_3 s^3 + ...,$$

For each of these polynomials, it is not necessary to compute the roots: checking whether all the roots of a polynomial $p(s)$ are in the $\varphi-$sector can be reduced to checking the stability of auxiliary polynomials

$$p(\exp(j\varphi)s) \cdot p(\exp(-j\varphi)s).$$

### D. *The fourth stage of the proposed method*

In the fourth (final) stage of the proposed method, we do the following:

- form the system of inequalities that describes all requirements for the desired control (these inequalities mainly come from applying Hurwitz criterion to the results of the third stage), and
- finally, determine the parameters of the desired control $\mathbf{R}(s)$.

### E. *How to make this method faster*

One possibility to sppeed up computation is to use simple sufficient conditions of robust stability of CLS instead of more computationally complicated necessary and sufficient stability conditions. There are also several other possibilities to reduce the computations time.

## V. Frequency Domain Methods and Algorithms

The possibility to use interval methods in interval domain is is based on the fact that Kharitonov's fundamental theorems enable us to find the precise values of upper and lower bounds for some frequency domain characteristics [6]. As a result, we can:

- compute the interval frequency domain characteristics for the plant and for the opened-loop system (OLS), and
- use these characteristics to evaluate the performance of the CLS.

For these methods, we have proposed an interval version of Nyquist stability criterion. This criterion forms the the foundation for the design of robust CLS controller. In addition to this criterion, we must analyze the connection between the interval frequency characteristics that correspond to different subsytems, and their responses (output signals).

This methodology also enables us to analyze systems with interval uncertainty by running computer simulations of appropriately chosen particular cases.

## VI. Software Implementation

We have implemented both methods of designing robust controllers for MIMO dynamic plants as a special software tool *Robustness-1*. This tool has already been applied o many different practical design problems.

*Robustness-1* can also help to describe the consequences of possible subsystem failures. For this application to be possible, we had to add multi-level control algorithms [7] to the above-described robust design methods.

## VII. The Main Application

This software tool has been successfully applied for the design of the robust control systems for:

- aircraft engines;
- airplanes themselves;
- power stations.

## References


[1] Yu. M. Gusev, V. N. Efanov, V. G. Krymsky, and V. Yu. Rutkowsky, "Analysis and Design of Linear Interval Dynamic Systems", *Izv. Acad. Nauk SSSR. Techn. kibernetika*, 1991, No. 1, pp. 3–24; No. 2, pp. 3–30 (in Russian).

[2] V. L. Kharitonov, "Asymptotic Stability of an Equilibrium Position of a Family of Systems of Linear Differential Equations", *Differential. Uravnen.*, 1978, Vol. 14, No. 11, pp. 1483–1485.

[3] V. I. Vasiliev, Yu. M. Gusev, V. N. Efanov, V. G. Krymsky, et al., **Multi-level Control for Dynamic Plants**, Moscow, Nauka Publ., 1987, 309 p. (in Russian).

[4] V. N. Efanov, V. G. Krymsky, and R. Z. Tlyashov, "The Control Algorithm Design for Multivariable Plant with Interval Parameters", *Izv. vuzov SSSR, Priborostroenie*, 1991, Vol. 34, No. 8, pp. 48–54 (in Russian).

[5] Y. C. Soh and Y. K. Foo, "Generalization of Strong Kharitonov Theorems for the Left Sector", *IEEE Transactions on Autom. Control*, 1990, Vol. 35, No. 12, pp. 1378–1382.

[6] B. R. Barmish, "Generalization of Kharitonov's Four-Polynomial Concept for Robust Stability Problems with Linearly Dependent Coefficient Perturbations", *IEEE Transactions on Autom. Control*, 1989, Vol. 34, No. 2, pp. 157–165.

[7] V. N. Efanov and V. G. Krymsky, "Multi-level Control Against Uncertainties and Failures Consequences", *Proceedings of PSAM-II International Conf. in San Diego (USA)*, Plenum, N.Y., 1994, Vol. 1, pp. 002.13–002.17.